Rice University School Mathematics Project
RUSMP High School Program
Course Syllabus
Summer 2009

Objectives

- Provide participants with mathematics activities, strategies, pedagogy, assessment, and materials for the secondary classroom which address NCTM Standards (1989, 1991, 1995, 2000) and state TEKS.
- Empower participants as teachers with enhanced understanding and appreciation of higher mathematical concepts.
- Influence the mathematics curriculum by modeling student-centered instruction and assessment using manipulatives and technology.
- Promote participation in professional and leadership roles for classroom teachers in the process of educational reform.

Mathematical Strands

- Patterns & Functions; Proportionality: Rate of Change & Slope; Parent Functions & Transformations; Linear Functions, Direct Variation, & Slope; Quadratic Functions; Exponential Functions; Systems of Linear Equations; Rational Functions; Asymptotes; Limits; Definition of Derivative; Parametric Equations; and Sequences and Series

Instructional focus

Central to the exploration of the above topics will be the common threads of problem solving, communication, reasoning and connections both within mathematics and with other subjects. Participants will be given opportunities to discover mathematical concepts through activities with graphing calculators, data collection devices, computers, and manipulatives in a variety of collaborative group settings. Assessment modes will include written, oral, artistic, kinesthetic, and other venues in which participants share their findings and their ideas.

Course Requirements

- Pre-Test/Post-Test – Participants will take a pre-test and post-test that will be used for program evaluation and research for RUSMP. The pre-test will not affect the course grade while the post-test will be part of the course assessment.
- Team Share. Pairs will present a 15 minute shortened version of a successful activity, including handouts and strategies for implementation.
- Learning Plan – Participants will develop and present a unit that is appropriate for a course they teach. Pairs present an overview of their learning plan and sample activity.
- Assignments – Assignments will reinforce mathematical concepts covered in class and will include selected professional readings that require a written response and/or in class discussions.
- Tests – There will be two intermediate tests as well as the pre-test and post-test to cover mathematical concepts and teaching strategies.
- Journal – Participants will keep a journal of their daily thoughts regarding activities, and experiences. While a certain length is not required, journals will be graded on the quality of your reflections. Journal entries for the week will be sent electronically by Friday at 11:59 p.m. to both of the master teachers. A late penalty will apply.

Class Duration:	June 8, 2009 – July 2, 2009	
	Mondays through Thursdays	
Class Hours:	Class:	8:30 am – 11:30 am
	Lunch:	11:30 am – 12:30 pm
	Class:	12:30 pm – 3:30 pm

Colloquia: Wednesdays, 10:30 – 11:30 in Multi-Purpose Room
Lunch will be provided.

Graduation Day: Thursday, July 2
Lunch will be provided.

Grading System:		
	Tests	35%
	Learning Plan	20%
	Assignments	15%
	Journal	10%
	Participation/Attendance	10%
	Team Share	10%

Learning Plan		
	Concept Selection	June 11
	Rough Draft	June 22
	Final Product	June 29 - July 1

Master Teachers: Julie Burnside jburnsid@houstonisd.org
Cedric French nfrench@houstonisd.org

Student Information Sheet

TO BE COMPLETED BY STUDENT (PLEASE PRINT)

Full Name (as it appears on records)

____

Last First Middle

Name you prefer to be called: ____________________

Classification: Freshman Sophomore Junior Senior Age: _______

3rd Period Teacher: ____________________ Room: ____________

Counselor: __________________ Advocate: ____________________

Assistant Principal: _______________

E-mail Address: ____

Last Math Class Taken: _________________ (Spring 2006)

Teacher: _________________

Other Math Classes You Are Taking this Year: _________________________

Complete this sentence: Math is _______________________________________.

Career Interests: ____

Hobbies / Extracurricular Activities / Jobs:

____

TO BE COMPLETED BY PARENT

My primary means of contacting you will be by your e-mail address given below. However, if one is not available, I can contact your home or work number. Please feel free to contact me about any special needs or circumstances I should know about.

Parent's/Guardian's Name: ___________________________

E-mail Address: ____

Home Phone: ___________________________

Work Phone: ___________________________

Mailing Address: ____

____

Please sign saying that you have read the syllabus for the course and are aware of the policies and procedures of the class. ___________________________

DATE	REASON	RESULT
	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer
	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer
	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer
	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer
	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer
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	Failing progress report Attendance Behavior	Talked to: E-mailed: Left message on machine No answer

Mr. French's Class Student Information

Period ______________ Date entered class ______________ Grade Level ________

Name: ____
Last First Middle

Prefer to be called: ______________________

Home phone number (list all numbers) ____________ 2nd Line ____________

Email Address: ____________________________________

Online Journal Address:____________________________________

Your Birth Date (mo./day /yr.) ___/___/___

Work phone #

Full Name of parent (s) 1) ______________________ ______________
2) ______________________ ______________

Others: ______________________ ______________

Parent/guardian is at home: Day Evening Day &Evening

Transportation to School: My Car HISD Bus Metro Parent/Relative Friend's Car

Your Schedule: List Course/Teacher/Room #

1. ______________________ 5. ______________________
2. ______________________ 6. ______________________
3. ______________________ 7. ______________________
4. ______________________ 8. ______________________

Dean: ______________________

List any school activities you are interested/involved in:
____
____

What interests you outside of school?
____
____

What are some songs/bands you like?
____
____

Do you have a job?______ Where do you work? ______________________

How many hours per week? ________

What did you like best about last year's math class (who was the teacher at what school)?
____
____

In the space below, write anything about yourself or your background about which you think I should be aware. Include anything you think I should know, but did not ask. (ex. I need to sit in the front because I have trouble seeing.)
____
____

Student Name:____

Parent Name:____

Parent Phone:____

Parent Email:____

DATE	COMMENTS

Can You Find Someone Who…

can state the Pythagorean Theorem	*has more than 2 pets at home*	*was not born in the United States*	*cannot believe they will be in class most of June!*	*has taught less than five years*
has taught more than ten years	*knows what* $\sin\frac{\pi}{2} =$	*is the oldest child in their family*	*is the youngest child in their family*	*is the middle child in their family*
has children under the age of five	*has been married for more than 10 years*	*has an interesting hobby*	*can state the quadratic formula*	*lives close by Pin Oak Middle School*
can name three conic sections	*can name four types of quadrilaterals*	*knows the formula for area of a trapezoid*	*knows the value of pi to at least five decimal places*	*lives far away from Pin Oak Middle School*

ANNOTATED LEARNING PLAN

<table>
<tr>
<td>Exploratory Activities
These are introductory "hands-on" activities to introduce students to a concept, e.g., a two-team mathematical Tic-Tac-Toe game that leads students to graph ordered pairs. These activities need to foster thinking and are not worksheet exercises.</td>
<td>Concept
A concept is an idea important in the main body of mathematics, e.g., multiplication, linear equations, area, slope. Concepts are used to organize instructional units. Concept-based organization encourages broad, rich units with connections among concepts within and across the mathematics curriculum.</td>
</tr>
<tr>
<td>Concept Development Activities
These are student-centered activities/problems aimed at providing students with exploration experiences to develop the concept in many situations so that formal learning and deep understanding can take place.</td>
<td>Materials and Resources
Examples: Algebra tiles, geoboards, Cuisenaire rods, etc., as well as, any necessary printed materials needed for the entire unit. Print materials should be referenced completely.</td>
</tr>
<tr>
<td>Procedural Knowledge and Basic Facts
Basic facts and algorithms form the computational component of the Learning Plan. They are integrated with conceptual development activities so that students develop fluency with valid and standard algorithms and basic facts. Textbook exercises and sets of procedure-based problems may be included here.</td>
<td rowspan="3">Originality and Creativity
Student Products

Written

Verbal

Kinesthetic

Visual

Student products provide opportunities for students to demonstrate their understanding of a concept in their own way. Students should select only one of these.</td>
</tr>
<tr>
<td>Assessment
These include formative and summative assessments. Teacher-made tests and alternative assessments (i.e., observations, student writing, portfolios, student self-evaluations, interviews, demonstration tasks) provide information about student learning and thinking, as well as, information upon which to base instructional decisions.</td>
</tr>
<tr>
<td>Related TEKS
These are the Texas Essential Knowledge and Skills throughout this concept.</td>
</tr>
</table>

Revised 04/15/08

LEARNING PLAN

<table>
<tr><td>Exploratory Activities</td><td>Concept</td></tr>
<tr><td>Concept Development Activities</td><td>Materials and Resources</td></tr>
<tr><td>Procedural Knowledge and Basic Facts</td><td rowspan="3">Originality and Creativity
Student Products
Written

Verbal

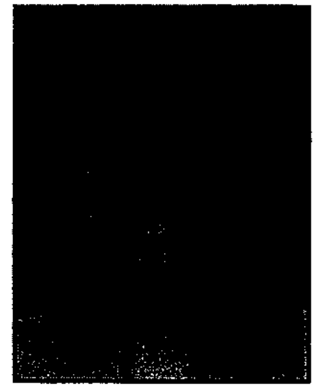
Kinesthetic

Visual</td></tr>
<tr><td>Assessment</td></tr>
<tr><td>Related TEKS</td></tr>
</table>

President's Message

"Try Harder!" Isn't the Answer

Cathy L. Seeley



Over the years, we have learned a lot about how students come to know mathematics and about how to teach for lasting learning. We have learned from looking at what other countries do, and we have learned by looking in our own backyard. Students learn challenging mathematics when they have opportunities to engage in problems and when a knowledgeable teacher guides their learning and helps them connect their classroom activities with the mathematics that underlies the activities.

Nevertheless, some administrators, policymakers, and communities have resisted supporting the use of promising teaching practices and materials, claiming that they are too difficult to implement or have no long-term data proving their effectiveness. Instead of looking for new approaches designed to make high-level mathematics accessible to all students, these groups call for a return to *traditional* methods, emphasizing skill development and calling for lectures by teachers as the primary means to accomplish it. In fact, we have considerable evidence from national and international assessments that the traditional approach has not served most of our students. Although I respect the motivation and commitment of all those involved in discussions on improving mathematics education in the United States, I question whether doing more of the same is the answer for the challenges we face. Where we once sought to educate a third of our students for study beyond high school, we now strive to educate *all* students to high levels. Today many more students will pursue some kind of postsecondary instruction than in the past. Even students who go straight into the workforce from high school now may need a basic understanding of algebra, physics, and electronic tools. Most of all, we know that the kinds of problems that employers, workers, and professionals now handle are often far more complex than those that commonly arose during the agricultural and industrial times that generated our traditional educational system. Simply stated, today's citizens and workers need a far deeper knowledge of mathematics and greater quantitative abilities than at any time in history.

I suggest that it is oversimplified, unrealistic, and unfair to try to raise students' achievement in mathematics simply by putting pressure on teachers to "try harder." To assume that teachers aren't already "trying hard enough" is grossly inaccurate. Across the board, teachers want students to achieve at high levels, and they do whatever they can to help them learn. But to accomplish the ambitious goal of a high-quality mathematics education for every student, educators, policymakers and communities will have to make significant, fundamental changes in the educational system, not just exhort teachers to "try harder."

We have to make hard choices about curricula, choosing to focus at each level on fewer topics and making a commitment to teach those topics for lasting understanding and learning. We need to invest in teachers not only through recruiting, but through mentoring new teachers, nurturing and respecting teachers at every stage of their careers, and offering teachers high-quality professional learning opportunities that help them continue to develop their mathematical knowledge and their understanding of teaching and learning mathematics. We need to allocate adequate resources for students, regardless of their school settings, and especially in high-poverty areas. This includes making sure that every student has access to a well-qualified teacher of mathematics. We need to teach in ways that engage students in doing mathematics and solving challenging problems instead of simply watching a teacher demonstrate mathematics. Finally, we need to make sure that our system offers opportunities for working across grades and levels and that the components of the system are well aligned with challenging and appropriate goals. These are important changes that go far beyond simply "trying harder." To do less is to deny teachers the tools, resources, and support that they need to make a real difference in the mathematics students learn.

When we do institute changes, we need to commit to following through and supporting those changes over time to allow teachers to refine what they do and to allow students to grow. We may implement different strategies to accomplish our goal of a high-quality mathematics education for every student. What we can't do is discard good programs just because they are difficult to implement or because we don't see immediate results. The most important thing we can do to serve our students is to listen to one another, learn from one another, and work together in true collaboration and sustained efforts toward the goal of a high-quality mathematics education for every single student.

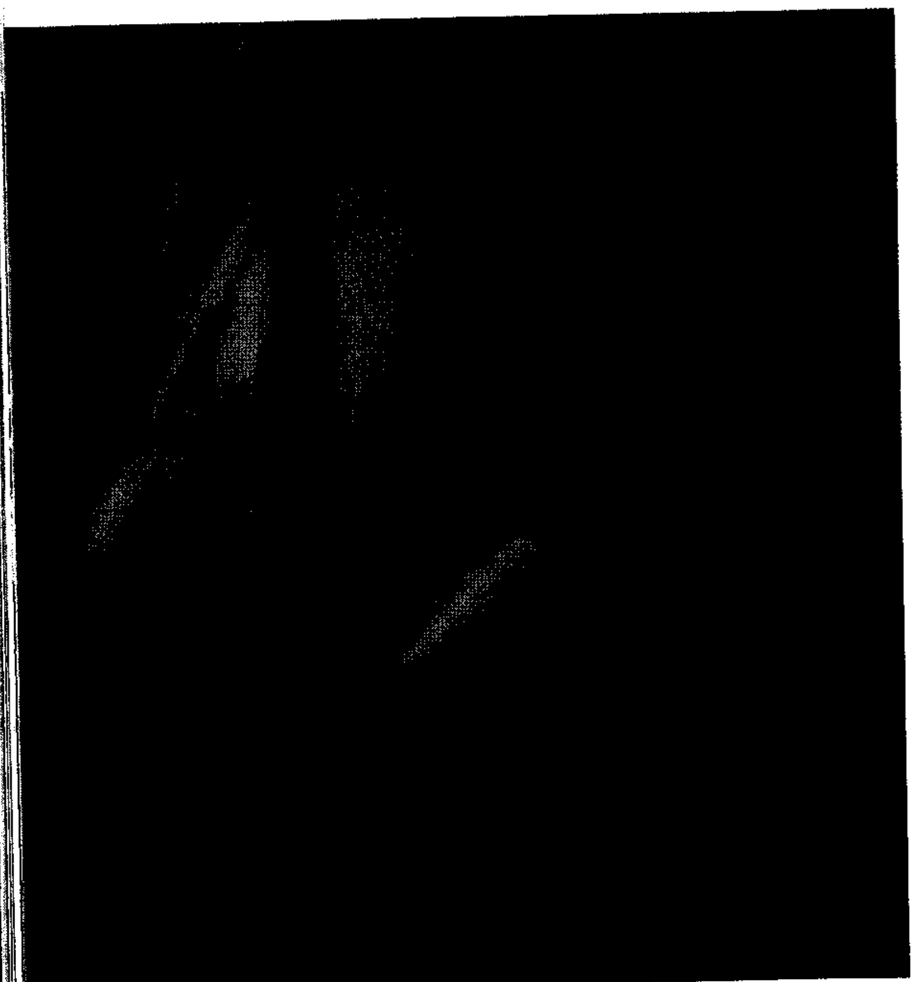
What are the most important changes that school systems should make to improve students' learning? What are the challenges that you face in making improvements in teaching and learning? Read the transcript of the May President's Chat at www.nctm.org/news/chat.htm to see how your colleagues answered these questions. Ω

Originally published in the *NCTM News Bulletin* (May/June 2005).

Critical Juncture Ahead!

Proceed with Caution to Introduce the Concept of Function

Gregorio A. Ponce



There are specific points in the teaching and learning of mathematics, critical junctures where students have great difficulty understanding a new and fundamental concept that affects to a great extent their success with future mathematical concepts and ideas. Take, for example, the concept of function: Its importance was established in mathematics centuries ago and still is acknowledged in national and state standards as a key topic of study. Kalchman and Koedinger (2005) affirm that "the new and very central concept introduced with function is that of a dependent relationship, the value of one thing depends on, is determined by, or is a function of another" (p. 352). Cunningham (2005) adds that "representing functions in multiple ways is critical to student understanding of functions and success in mathematics" (p. 73). A key goal, Eisenberg (1992) states succinctly, is "to develop in students *a sense for functions*" (p. 154). Numerous articles have been written offering strategies that target such difficulties (Bossé 2005; Davis 2005; Fernandez 2005; Kersaint 2006; Rivera and Becker 2005; Van Dyke and White 2004). Incorporating new instructional strategies into the classroom, however, is not always a straightforward matter.

One big challenge that teachers face is finding the right balance: Can students achieve conceptual understanding and procedural fluency (Fuson, Kalchman, and Bransford 2005) while being

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Coordinate system	Plotting points	Deciding if two lines are parallel	Defining linear functions	Deciding if a set of ordered pairs is a function
Solving direct variation problems	Defining a function	Finding coordinates of points	Graphing using a point and the slope	Defining quadratic equations
Equations of horizontal lines and their graphs	Finding the domain of graphs	Finding joint variation equations	Finding values of a function	Graphing linear equations
Using a t-table to graph an equation	Defining the slope/intercept form of an equation	Fitting equations to real data	Defining perpendicular lines	Identifying the quadrants given specific ordered pairs
Finding the range of a function	Deciding if two lines are perpendicular	Graphing linear equations using intercepts	Writing an equation in point-slope form	Finding an equation given its graph
Deciding when a graph represents a function	Deciding if a point is a solution of an equation	Finding the domain of a function	Defining intercepts	Finding corresponding solutions
Writing a linear equation in standard form	Defining slope	Finding the intercepts of quadratic equations	Finding the slope of a line given an equation	Defining a relation
Function notation	Finding the y-intercept of an equation	Deciding if a relation is a function	Finding inverse variation equations	Using the vertical line test
Finding the equation of a line using two points	Graphing quadratic equations	Finding the slope given two points	Ways to represent a function	Equations of vertical lines and their graphs
Finding combined variation equations	Solving for y when working with linear equations	Solving word problems with linear equations	Finding the equation of a line using the slope and a point	Graphing using the slope and the y-intercept
Defining the constant of variation	The domain and range of a quadratic equation	Graphing functions	Using slope to decide if two lines are perpendicular	Writing a quadratic equation in standard form

Fig. 1 Typical topics covered in an algebra course related to the concept of function

prepared to excel in examinations that test comprehension of state/district content standards? A cursory review of the typical exercises and topics covered in a traditional algebra textbook associated with the concept of function (**fig. 1**) clearly illustrates the sheer volume of information that needs to be taught and learned in the classroom and also helps explain why so many teachers are hard pressed to find such a balance with their students. One way to address this issue is to emphasize key mathematical concepts and their corresponding core ideas (Ma 1999). In this article, I share how language and communication can help establish the core ideas that define a function and then suggest activities to introduce the concept of function through the use of language and the core ideas.

A rule that assigns to each number x (the input) a single value y (the output)	A set of ordered pairs that assigns to each x-value exactly one y-value
A relation for which to each domain value there corresponds exactly one range value	A rule or correspondence that assigns to each element of the set X one and only one element of the set Y
A set of ordered pairs in which each first component in the ordered pairs corresponds to exactly one second component	A relation that matches each element of a first set to an element in a second set in such a way that no element in the first set is assigned two different elements in the second set
A relation in which for each value of x there is a unique value of y; x is an independent variable, y is the dependent variable.	A rule that takes certain numbers as inputs and assigns to each a definite output
A special type of relation in which no two ordered pairs have the same first coordinate and different second coordinates	A set of operations that are performed on each value that is put into it and results in one answer

Fig. 2 Mathematical definitions of a function

WHAT DOES *f*(x) MEAN ANYWAY?

When asked to share what $f(x) = 2x + 30$ means to them, mathematics teachers will tap into their prior knowledge and expertise and state that it is a function, or a rule to find the values of y, or a linear function with slope of 2 and y-intercept of 30, or a representation of a dependent relationship, and so on. Most teachers will also readily agree that the majority of their students would not have the same insights about the function. The issue thus becomes how to introduce the concept of function to students in a way that taps into *their* prior knowledge and experiences. If, according to Usiskin (2005), "learning algebra should be no more difficult than learning a new language" (p. 13), then the first step in this process is to discuss the everyday meaning of the word *function* because "this provides a base on which to build a connection to formal mathematical language" (NCTM 2000, p. 63).

Setting mathematics aside for a moment, what does the word *function* mean to you in everyday language? What is your function (purpose) as a teacher? Do you find it helpful to change your function (role) as you teach mathematics to your students? Does your school have a special function (social event) to recognize the mathematical achievement of your students at the end of each school year? Is there a particular function (task) that your students need to complete in the first five minutes of class? In all four cases, the word *function* is used as a noun; it is some*thing*. On the other hand, one could have asked you to think about the time of day that your students function (work) best when learning mathematics, or perhaps how students function (perform) when presenting their solution strategies in front of the whole class. Do your students function (act) differently when they are taught by a substitute teacher? Taken as a verb, a function *does* something. One could say, then, that in mathematics one studies what this *thing* called a function looks like (graphically, numerically, symbolically) to shed more light on what it *does* (establishes a dependent relationship). It is important to note that considering the word as a noun and as a verb also helps to identify the corresponding core ideas of a function.

Setting mathematics back at center stage: How does one decide whether the definitions in **figure 2** correctly define the concept of function? What helps you sift through such varied definitions of the same concept? Using the discussion above as a starting point, if a function is a thing, then the definitions convey what it is: a set of operations, a relation, a rule, a set of ordered pairs, a correspondence. If a function does something, then the list clarifies what it does: it assigns, it matches, it corresponds, it performs, it operates, it relates. There are three other instances where the definitions describe the same idea but with different words. One group of words includes the terms *input, x, first, x-value, domain,* and *independent;* by implication, the related terms *output, y, second, y-value, range,* and *dependent* would form another group. The final group of words, which at times have to be phrases, includes *single, definite, one and only one, exactly one, unique,* and *the first element cannot have different second elements.* The key here is to illustrate how language connects the everyday and mathematical meanings of the word *function* and how language helps bring to the forefront the core ideas that define this important concept. The intent is to help make obvious for students what is self-evident to many mathematics teachers.

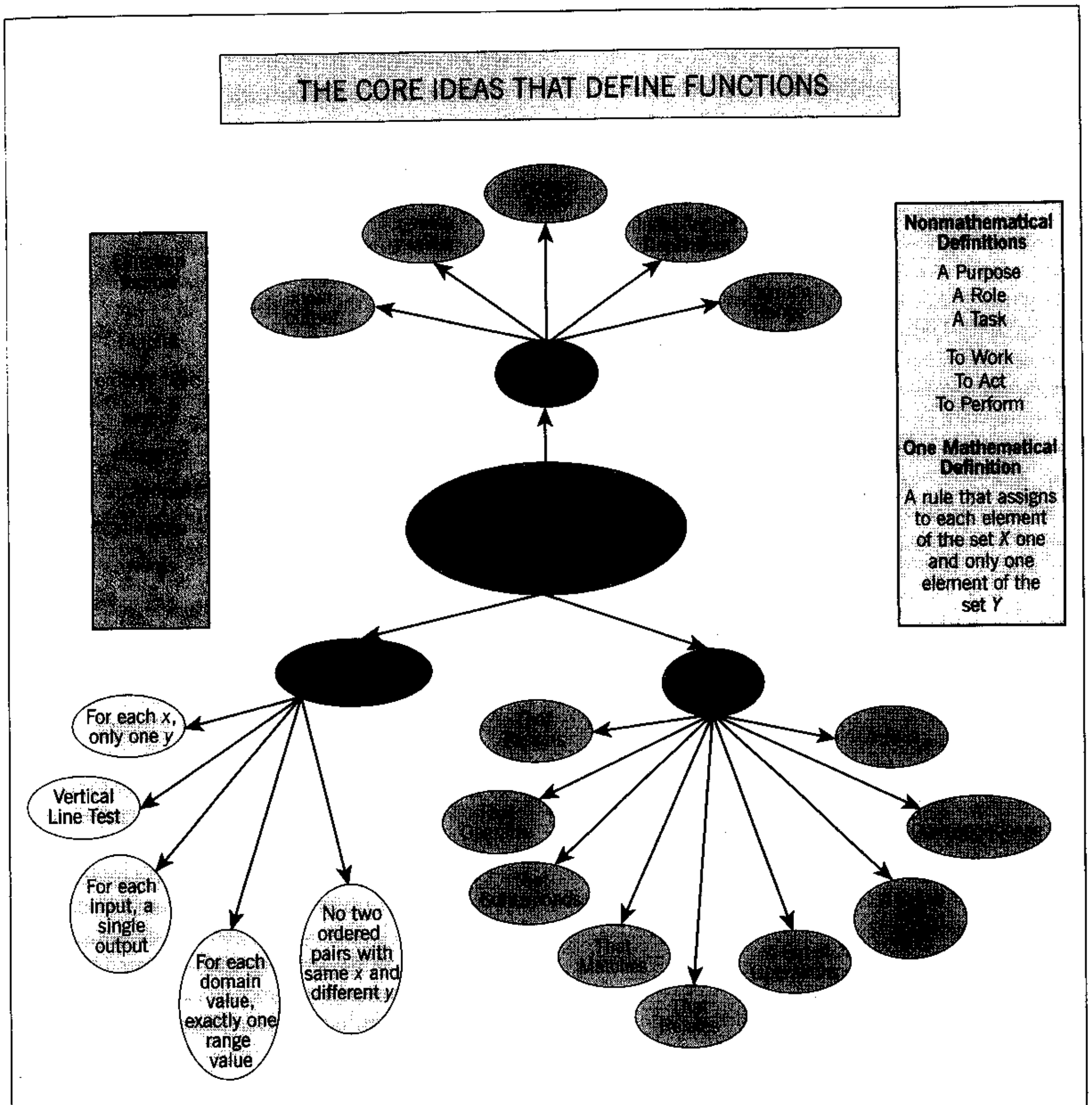

Fig. 3 Core ideas that define functions

THE CORE IDEAS THAT DEFINE A FUNCTION

The following approach has been used successfully with students to introduce the core ideas of a function and to illustrate the usefulness of a function outside the classroom. Given the significance of communication as a standard of mathematics (NCTM 2000), in particular as it relates to the use of language that keeps the target audience in mind, e.g., algebra students, I have named each core idea so that it makes sense to students who are starting to learn about functions. Thus, a bridge is formed to the mathematical meaning and application of those same ideas. This approach, as presented in the classroom to students, is more of a conversation driven by questions than a lecture, because it is "seeing the connections between the different points of view that is important, not simply seeing the concept represented in a different context" (Eisenberg 1992, p. 159).

First core idea

When was the last time that you went to a gas station because your car, or your parent's car, was running low on fuel? To fill up the tank you need to pay for the gas. To buy the gas, you need money. Nothing can occur unless there is money and gasoline. These two things, money and gasoline, point to the first core idea of a function: You need *two groups* of something that, preferably, can be represented with numbers. Each of the two groups represents a set or collection of objects. Typically, the first set is called the domain, and the second set is called the range. Keep in mind, though, that there are other names, such as input and output, which also refer to the same two sets (shown in orange in **fig. 3**). Height and age, hours and wages, distance and speed, are other examples associated with this core idea.

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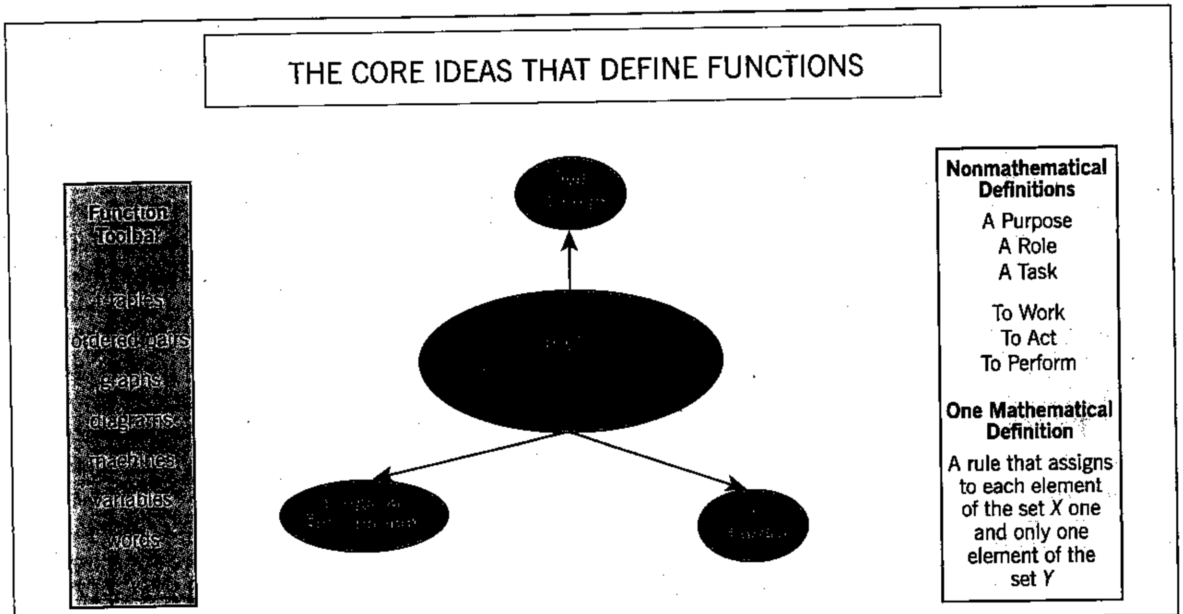

Fig. 4 Core ideas that define functions

The teacher then showed the class a function machine—just the machine. Students were told to look for a pattern between the numbers that were put into the machine and the numbers that came out. A 5 became an 11, a 6 became a 12, a 9 became a 15, an 8 would become a what? Almost immediately, a student said that the machine was adding 6, and the teacher asked the class to decide if this is in fact the case. When everyone agreed, the teacher distributed a copy of the concept map as shown in **figure 4**, led a whole-class discussion on how this example related to the core ideas, and eventually came up with **figure 5**. After the class worked through a few more machine examples, students were told to come up with their own function machines for the next class session. The next day, students were ready to share their machines with the class, and the teacher introduced the many ways in which one can represent a function. In reflecting about this experience, the teacher concluded that students were much more engaged and interested in learning about functions.

One can adapt this activity to provide students with various nonmathematical definitions **(fig. 6)** of the word *function* to facilitate a similar exchange of ideas on its everyday meanings. In either case, the point is to establish the link to students' prior knowledge by having them think about and discuss what the word *function* means in everyday language in order to begin their transition toward a better understanding of what a function is, looks like, and does, *mathematically*. In fact, terms such as *opposite, intercept, variable, slope,* and *limit,* in addition to those identified in *Principles and Standards for School Mathematics* (NCTM 2000, p. 63), would also work well in this activity. Schwartzman's mathematical dictionary (1994) and a regular English dictionary

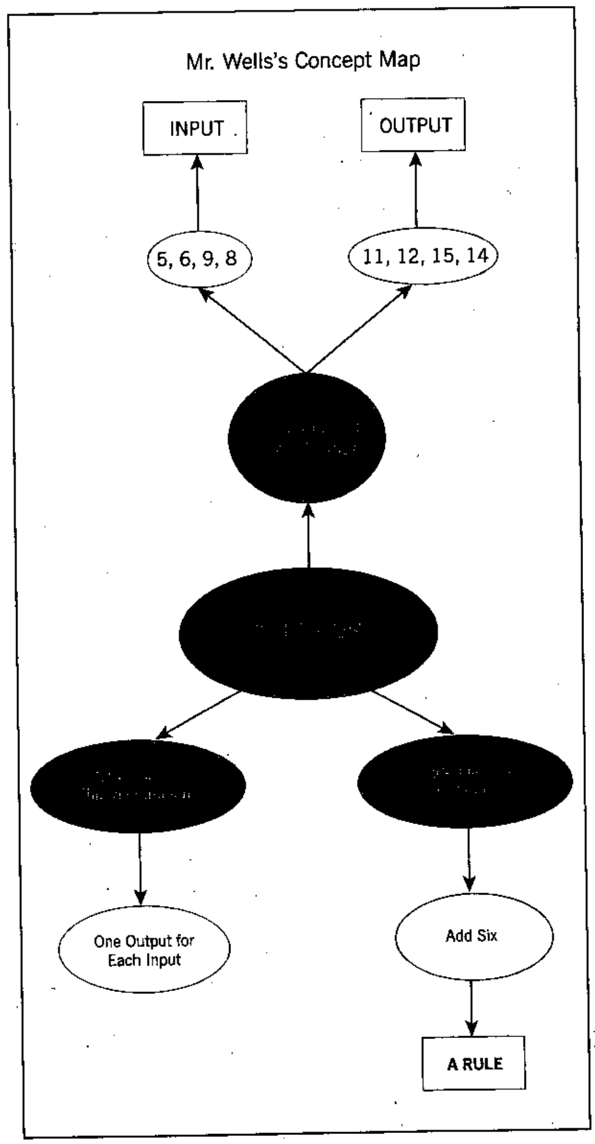

Fig. 5 Concept map used in a prealgebra class

Second core idea

The act of purchasing gasoline is a process that defines a relationship between these two sets, money and gas. In this example, a gallon of gasoline costs $3.00, so $18.00 can buy 6 gallons, $27.00 can buy 9 gallons, and $36.00 can buy 12 gallons. How the numbers relate to each other illustrates the second core idea of a function: *A pattern* exists between the two groups of numbers or objects (shown in blue in **fig. 3**). To uncover a pattern means to find out how numbers in the domain, the first set, are related or paired with numbers in the range, the second set. There are many mathematical tools, such as t-tables, ordered pairs, equations, and graphs, to help make the pattern, or relationship, more obvious and understandable (shown in purple in **fig. 3**). Some patterns are so common that they have specific names, e.g., linear, exponential, and quadratic.

Third core idea

Finally, on any given day, if two people were to drive up in their own cars to fill up their tanks and each paid $18.00, one would expect that both would receive the same amount of gasoline. That is, regardless of who is purchasing the gas, $18.00 should buy, according to this example, 6 and only 6 gallons. If this is the case, then the relationship addresses the third core idea of a function: The pattern between the two groups of numbers must meet a special requirement—that is, each number from the domain is matched or assigned to one and only one number from the range (shown in yellow in **fig. 3**). If this condition is not met, then one does not have a function, one has a relation.

In order to bring closure to this introduction on functions, a PowerPoint presentation (available at www.ivcampus.sdsu.edu/math_ed/index_files/Functions_PPS.pps) helps facilitate a summative discussion about the core ideas and their usefulness. A key advantage to those who use this type of medium is that it provides an ability to describe, in a dynamic manner, the same function with numbers, visuals, graphics, and equations. This can help students "understand that these are different ways of describing the same relationship" (Kalchman and Koedinger 2005, p. 352). These representations also provide students with more ways in which to understand the nature of functions. This introductory activity is likely to raise more questions than answers from students. For example, they will want to know how to find the equation, thus setting the stage for discussions about functions, their usefulness, and different representations. Then again, this is the point of the activity: to create in students an initial sense of wonder about and appreciation for the concept of function.

THE CRITICAL JUNCTURE: INTRODUCING THE CONCEPT OF FUNCTION

Too often, students have little sense of what to do when given functions to explore, whether they are presented in word problems or numerically, graphically, or symbolically. When teachers help students identify the core ideas within the problems, students can have a sense of what to look for as they sift through the details of the scenarios and/or the specific skills in their textbooks. In other words, if the new concept is a dependent relationship, students need to realize that it takes two sets to form a relationship and those sets and the pattern relating them need to be readily identified. Understanding the connections provides students with insights into the nature of the relationship between the two sets (second core idea); and this type of relationship must satisfy a special requirement in order to be considered a function (third core idea).

The core ideas of a function can also help teachers restore for students some of the connections and links that are lost in the presentation in textbooks, in which knowledge "has to be taken apart and ordered sequentially . . . into a large number of isolated bits of knowledge" (Eisenberg 1992, p. 168). According to Donovan and Bransford (2005), the key to restoring such connections is recognizing that "memory of factual knowledge is enhanced by conceptual knowledge, and conceptual knowledge is clarified as it is used to help organize constellations of important details" (p. 7). In this section I share how some mathematics teachers use language and the core ideas to introduce the concept of function to students. I also offer some adaptations of their ideas, inviting readers to take them as a starting point to explore how the core ideas can help students improve their sense of what to do when working with functions.

Introducing function *to prealgebra students*

A prealgebra teacher decided to use these ideas to introduce students to the concept of function. At the start of the activity, students were asked to find as many meanings as they could for the word *function,* using the Internet, dictionaries, and thesauruses, and talking to adults, such as their parents or teachers, to complete the assignment. The next day, students shared what the word meant to them. As they shared the different meanings of the word, the teacher would interject, when appropriate, how some meanings of the word implied action—that is, to function is to do something. To transition the discussion, the teacher showed them **figure 4** and explained how this concept map would help students see the relationships among the mathematical ideas about functions. Then the teacher showed the PowerPoint presentation on functions as a way to introduce and clarify the meaning of the three core ideas.

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If something functions, it works.	A purpose	A role
A formal social gathering	To perform	An action or activity proper to a person or thing
The purpose for which something is designed	A ceremonious public or social occasion or gathering	To work
To operate	To serve	To perform a function
The acts or operations expected of a person or thing	A job	A task
An action	A behavior	To operate in an expected or proper manner
An activity	To act	To do

Fig. 6 Nonmathematical definitions

are useful in preparing introductions of these concepts. Another adaptation of the activity is to make a class bulletin board of **figure 3**, in addition to the handout, to help students internalize a visual representation of the concept as it is developed over time.

Introducing function *to algebra students*

An algebra teacher introduced the concept of functions by leading a class discussion about the everyday meaning of the word and using the PowerPoint presentation to reinforce the core ideas. The class was shown a picture of a boy dropping a ball from a height of 48 inches. Students were told that after each bounce, the ball would reach half its previous height. The task for the students was to make a table of entries and a graph and to find an equation that would help them predict the height of the ball given the number of bounces the ball had completed. Before letting his students begin their work, the teacher facilitated a whole-class discussion to identify how the core ideas of a function applied to this problem. **Figure 7** reflects the results of the discussion. Afterward, groups were formed, and students began to work on the task. While walking around the room observing student work, the teacher was able to see the different ways, some right and some wrong, in which students answered the questions. After some time, some students were asked to present their work to illustrate that there are different ways to represent the same kind of information. This was also an opportunity to clarify some misconceptions and address some mistakes. Most of the discussion, however, revolved around trying to find the right function to represent the pattern—not an easy task, since it was an exponential function. Once the function was established, students used it to find the height for different numbers of bounces. The algebra teacher, like the prealgebra teacher, was encouraged by students'

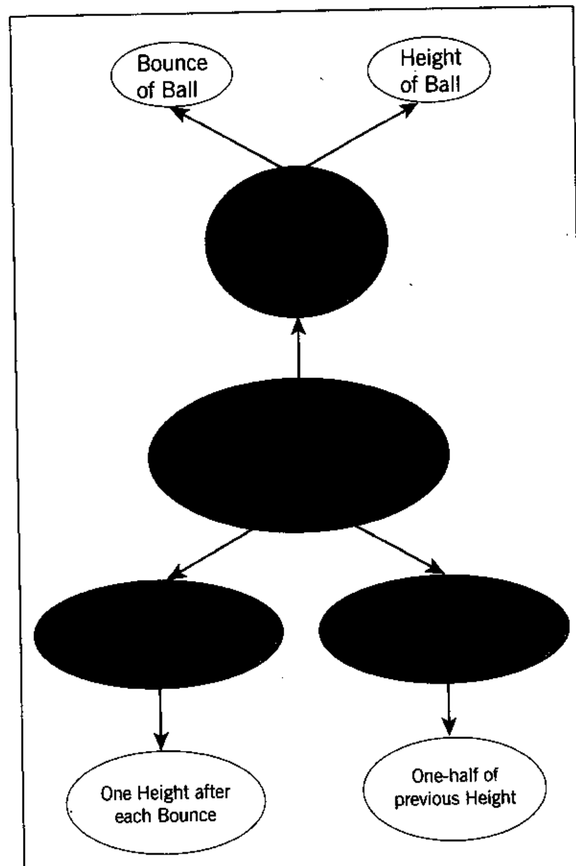

Fig. 7 Class concept map generated by a discussion in an algebra class

reactions and the results of using the core ideas to organize instruction and learning.

The challenge for the algebra teacher is to find a way to help students recognize that the same concepts and skills covered in the text are also being addressed through word problems, like the one about the bouncing ball. One adaptation, then, can be to make clearer in the concept map the relationship between the wording in the problem and the

conventional mathematical notation used to express those same ideas. For instance, as students presented their graphs, the discussion could have included how the x-axis represented the number of bounces for the ball and that the y-axis represented the height of the ball. The concept map could thus be updated to show some of the more familiar mathematical notation, *e.g.*, x and y, next to the name of the two groups. Consequently, when the section on the Cartesian coordinate system begins, terms such as *abscissa, ordinate,* and *ordered pairs* can be linked to, and used to expand upon, the first core idea of a function. Also, through the use of a class bulletin board with the core ideas, the teacher can begin to form in students a collective understanding of the relationships among the details of the scenario, the concept of function, and the mathematics topic of the day.

TAKE 2: WHAT DOES $f(x)$ MEAN ANYWAY?

Technically, $f(x) = 2x + 30$ tells students all there is to know about the function. The notation is intended to convey to the student what this *thing* is *doing* to numbers from the domain. To help make this less abstract and more meaningful to students, ask students to figure out how much money they would spend at an amusement park if the entrance fee is \$30 and the price for each ride is \$2. Students will see that the amount of money spent will depend on the number of rides that they take—*two groups:* money spent and rides taken. Asking students to find out the amount of money spent for any particular number of rides taken will give them a better sense of what this function is doing. Also, guiding them to represent their thinking through the use of a t table or ordered pairs, e.g., (0, 30), (1, 32), (2, 34), and then graphing those points, will give students additional insights about *the pattern*: a line that rises from left to right or numbers that increase by two each time. A clearer picture begins to form regarding the domain and the range, as well as how the function meets the special requirement: the number of rides taken determines one and only one amount to spend.

Of particular importance is to link such insights to the conventional notation and technical language of functions. That is, the notation $f(x) = 2x + 30$ gives a name for this *thing* called a function, i.e., f; it explicitly identifies the variable for the domain of the function, x; it represents the range of the function, $f(x)$; and it shows what the function looks like through a set of operations represented symbolically by $2x + 30$. This set of operations also clarifies what the function is *doing* to establish the dependent relationship between the domain and the range. **Figure 8** shows how a concept map centered on the core ideas can clarify for students the meaning of $f(x) = 2x + 30$.

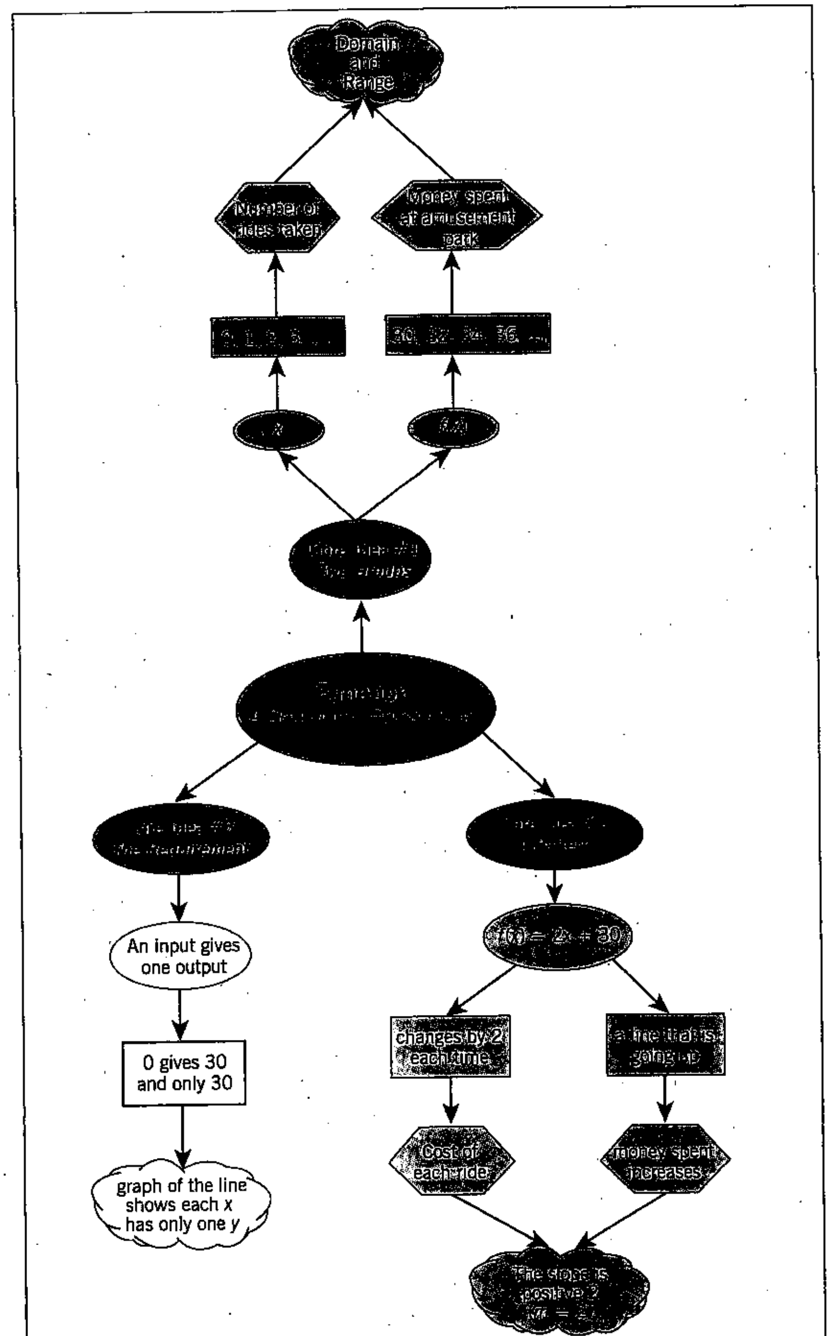

Fig. 8 How a concept map can clarify the meaning of $f(x) = 2x + 30$

CONCLUSION

By studying words and definitions through different sources, one can form a basic idea of what the concept requires and what it is meant to accomplish.
—J. Santillanes, algebra teacher

In many regards, the statement by Mr. Santillanes encapsulates the essence of this article because "mathematics requires careful reasoning about precisely defined objects and concepts" (Ball et al. 2006, par. 8), particularly when the same word, like the word *function,* can have many different and valid mathematical definitions. Accordingly, through the use of language

one can sift through these multiple definitions to uncover the core ideas that define the concept of function. To be clear, one goal of this article is to illustrate how the use of everyday language can serve as an effective conduit to help students learn the very technical and powerful language of mathematics. The intent is to raise the level of sophistication in which students communicate mathematically as their understanding of and experience with a concept evolves. From this vantage point, language becomes a bridge, rather than a barrier, to student success with mathematics.

Another goal of this article is to posit that students develop a better sense for functions (Eisenberg 1992) when teachers begin with the core ideas. Using these ideas to organize lessons and to facilitate whole-class discussions can be the framework to meet such a goal because "when one understands Big ideas, mathematics is no longer seen as a set of disconnected concepts, skills, and facts. Rather, mathematics becomes a coherent set of ideas" (Charles 2005, p. 10). Also, "using concepts to organize information stored in memory allows for much more effective retrieval and application" (Donovan and Bransford 2005, p. 7), which can be of great help to students as they try to solve problems on their own. Therefore, from this vantage point, the core ideas can become anchors for both student and teacher as they engage in the teaching and learning of mathematics.

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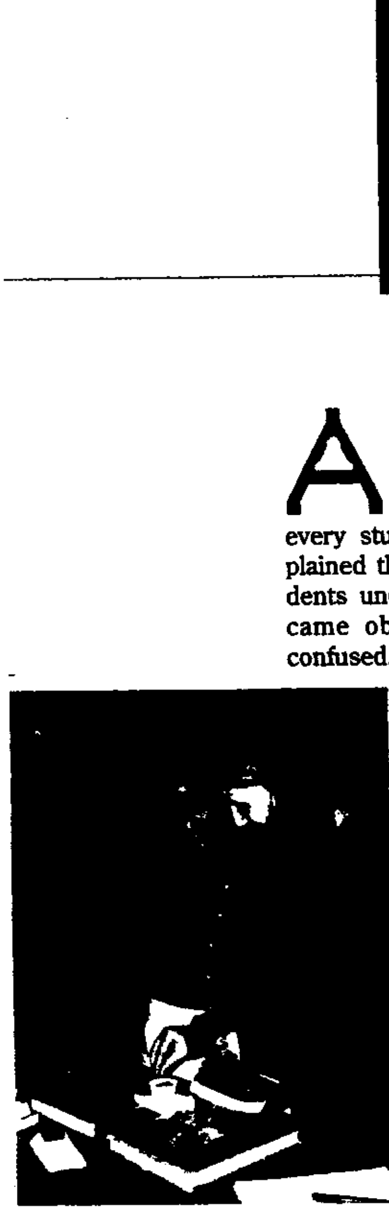
16

Never Say Anything a Kid Can Say!

STEVEN C. REINHART

AFTER EXTENSIVE PLANNING, I PRESENTED what should have been a masterpiece lesson. I worked several examples on the overhead projector, answered every student's question in great detail, and explained the concept so clearly that surely my students understood. The next day, however, it became obvious that the students were totally confused. In my early years of teaching, this situation happened all too often. Even though observations by my principal clearly pointed out that I was very good at explaining mathematics to my students, knew my subject matter well, and really seemed to be a dedicated and caring teacher, something was wrong. My students were capable of learning much more than they displayed.



Implementing Change over Time

THE LOW LEVELS OF ACHIEVEMENT of many students caused me to question how I was teaching, and my search for a better approach began. Making a commitment to change 10 percent of my teaching each year, I began to collect and use materials and ideas gathered from supplements, workshops, professional journals, and university classes. Each year, my goal was simply to teach a single topic in a better way than I had the year before.

Before long, I noticed that the familiar teacher-centered, direct-instruction model often did not fit well with the more in-depth problems and tasks that I was using. The information that I had gathered also suggested teaching in nontraditional ways. It was not enough to teach better mathematics; I also had to teach mathematics better. Making changes in instruction proved difficult because I had to learn to teach in ways that I had never observed or experienced, challenging many of the old teaching paradigms. As I moved from traditional methods of instruction to a more student-centered, problem-based approach, many of my students enjoyed my classes more. They really seemed to like working together, discussing and sharing their ideas and solutions to the interesting, often contextual, problems that I posed. The small changes that I implemented each year began to show results. In five years, I had almost completely changed both *what* and *how* I was teaching.

The Fundamental Flaw

AT SOME POINT DURING THIS METAMORPHOSIS, I concluded that a fundamental flaw existed in my teaching methods. When I was in front of the class demonstrating and explaining, I was learning a great deal, but many of my students were not! Eventually, I concluded that if my students were to ever really learn mathematics, *they* would have to do the explaining, and *I*, the listening. My definition of a good teacher has since changed from "one who explains things so well that students understand" to "one who gets students to explain things so well that they can be understood."

Getting middle school students to explain their thinking and become actively involved in classroom discussions can be a challenge. By nature, these

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students are self-conscious and insecure. This insecurity and the effects of negative peer pressure tend to discourage involvement. To get beyond these and other roadblocks, I have learned to ask the best possible questions and to apply strategies that require all students to participate. Adopting the goals and implementing the strategies and questioning techniques that follow have helped me develop and improve my questioning skills. At the same time, these goals and strategies help me create a classroom atmosphere in which students are actively engaged in learning mathematics and feel comfortable in sharing and discussing ideas, asking questions, and taking risks.

Questioning Strategies That Work for Me

ALTHOUGH GOOD TEACHERS PLAN DETAILED lessons that focus on the mathematical content, few take the time to plan to use specific questioning techniques on a regular basis. Improving questioning skills is difficult and takes time, practice, and planning. Strategies that work once will work again and again. Making a list of good ideas and strategies that work, revisiting the list regularly, and planning to practice selected techniques in daily lessons will make a difference.

Students feel comfortable sharing and discussing ideas

Create a plan. The following is a list of reminders that I have accumulated from the many outstanding teachers with whom I have worked over several years. I revisit this list often. None of these ideas is new, and I can claim none, except the first one, as my own. Although implementing any single suggestion from this list may not result in major change, used together, these suggestions can help transform a classroom. Attempting to change too much too fast may result in frustration and failure. Changing a little at a time by selecting, practicing, and refining one or two strategies or skills before moving on to others can result in continual, incremental growth. Implementing one or two techniques at a time also makes it easier for students to accept and adjust to the new expectations and standards being established.

1. Never say anything a kid can say! This one goal keeps me focused. Although I do not think that I have ever met this goal completely in any one day or even in a given class period, it has forced me to develop and improve my questioning skills. It also sends a message to students that their participation is essential. Every time I am tempted to tell students something, I try to ask a question instead.

2. Ask good questions. Good questions require more than recalling a fact or reproducing a skill. By asking good questions, I encourage students to think about, and reflect on, the mathematics they are learning. A student should be able to learn from answering my question, and I should be able to learn something about what the student knows or does not know from her or his response. Quite simply, I ask good questions to get students to think and to inform me about what they know. The best questions are open-ended, those for which more than one way to solve the problem or more than one acceptable response may be possible.

3. Use more process questions than product questions. Product questions—those that require short answers or a yes or no response or those that rely almost completely on memory—provide little information about what a student knows. To find out what a student understands, I ask process questions that require the student to reflect, analyze, and explain his or her thinking and reasoning. Process questions require students to think at much higher levels.

4. Replace lectures with sets of questions. When tempted to present information in the form of a lecture, I remind myself of this definition of a lecture: "The transfer of information from the notes of the lecturer to the notes of the student without passing through the minds of either." If I am still tempted, I ask myself the humbling question "What percent of my students will actually be listening to me?"

5. Be patient. Wait time is very important. Although some students always seem to have their hands raised immediately, most need more time to process their thoughts. If I always call on one of the first students who volunteers, I am cheating those who need more time to think about, and process a response to, my question. Even very capable students can begin to doubt their abilities, and many eventually stop thinking about my questions altogether. Increasing wait time to five seconds or longer can result in more and better responses.

Good discussions take time; at first, I was uncomfortable in taking so much time to discuss a single question or problem. The urge to simply tell my students and move on for the sake of expedience was considerable. Eventually, I began to see the value in what I now refer to as a "less is more" philosophy. I now believe that all students learn more when I pose a high-quality problem and give them the necessary time to investigate, process their thoughts, and reflect on and defend their findings.

explain your thinking to me?" As soon as I tell a student that the answer is correct, thinking stops. If students explain their thinking clearly, I ask a "What if?" question to encourage them to extend their thinking.

Participation is not optional! I remind my students of this expectation regularly. Whether working in small groups or discussing a problem with the whole class, each student is expected to contribute his or her fair share. Because reminding students of this expectation is not enough, I also regularly apply several of the following techniques:

No one is finished until all can explain the solution

1. Use the think-pair-share strategy. Whole-group discussions are usually improved by using this technique. When I pose a new problem; present a new project, task, or activity; or simply ask a question, all students must think and work independently first. In the past, letting students begin working together on a task always allowed a few students to sit back while others took over. Requiring students to work alone first reduces this problem by placing the responsibility for learning on each student. This independent work time may vary from a few minutes to the entire class period, depending on the task.

After students have had adequate time to work independently, they are paired with partners or join small groups. In these groups, each student is required to report his or her findings or summarize his or her solution process. When teams have had the chance to share their thoughts in small groups, we come together as a class to share our findings. I do not call for volunteers but simply ask one student to report on a significant point discussed in the group. I might say, "Tanya, will you share with the class one important discovery your group made?" or "James, please summarize for us what Adam shared with you." Students generally feel much more confident in stating ideas when the responsibility for the response is being shared with a partner or group. Using the think-pair-share strategy helps me send the message that participation is not optional.

A modified version of this strategy also works in whole-group discussions. If I do not get the responses that I expect, either in quantity or quality, I give students a chance to discuss the question in small groups. On the basis of the difficulty of the question, they may have as little as fifteen seconds or as long as several minutes to discuss the question with their partners. This strategy has helped improve discussions more than any others that I have adopted.

2. If students or groups cannot answer a question or contribute to the discussion in a positive way, they must ask a question of the class. I explain that it is all right to be confused, but students are responsible for asking questions that might help them understand.

3. Always require students to ask a question when they need help. When a student says, "I don't get it," he or she may really be saying, "Show me an easy way to do this so I don't have to think." Initially, getting students to ask a question is a big improvement over "I don't get it." Students soon realize that my standards require them to think about the problem in enough depth to ask a question.

4. Require several responses to the same question. Never accept only one response to a question. Always ask for other comments, additions, clarifications, solutions, or methods. This request is difficult for students at first because they have been conditioned to believe that only one answer is correct and that only one correct way is possible to solve a problem. I explain that for them to become better thinkers, they need to investigate the many possible ways of thinking about a problem. Even if two students use the same method to solve a problem, they rarely explain their thinking in exactly the same way. Multiple explanations help other students understand and clarify their thinking. One goal is to create a student-centered classroom in which students are responsible for the conversation. To accomplish this goal, I try not to comment after each response. I simply pause and wait for the next student to offer comments. If the pause alone does not generate further discussion, I may ask, "Next?" or "What do you think about _________'s idea?"

5. No one in a group is finished until everyone in the group can explain and defend the solution. This rule forces students to work together, communicate, and be responsible for the learning of everyone in the group. The learning of any one person is of little value unless it can be communicated to others, and those who would rather work on their own often need encouragement to develop valuable communication skills.

Share with students reasons for asking questions. Students should understand that all their statements are valuable to me, even if they are incorrect or show misconceptions. I explain that I ask them questions because I am continuously evaluating what the class knows or does not know. Their comments help me make decisions and plan the next activities.

Teach for success. If students are to value my questions and be involved in discussions, I cannot use questions to embarrass or punish. Such questions accomplish little and can make it more difficult to create an atmosphere in which students feel comfortable sharing ideas and taking risks. If a student is struggling to respond, I move on to another student quickly. As I listen to student conversations and observe their work, I also identify those who have good ideas or comments to share. Asking a shy, quiet student a question when I know that he or she has a good response is a great strategy for building confidence and self-esteem. Frequently, I alert the student ahead of time: "That's a great idea. I'd really like you to share that with the class in a few minutes."

Be nonjudgmental about a response or comment. This goal is indispensable in encouraging discourse. Imagine being in a classroom where the teacher makes this comment: "Wow! Brittni, that was a terrific, insightful response! Who's next?" Not many middle school students have the confidence to follow a response that has been praised so highly by a teacher. If a student's response reveals a misconception and the teacher replies in a negative way, the student may be discouraged from volunteering again. Instead, encourage more discussion and move on to the next comment. Often, students disagree with one another, discover their own errors, and correct their thinking. Allowing students to listen to fellow classmates is a far more positive way to deal with misconceptions than announcing to the class that an answer is incorrect. If several students remain confused, I might say, "I'm hearing that we do not agree on this issue. Your comments and ideas have given me an idea for an activity that will help you clarify your thinking." I then plan to revisit the concept with another activity as soon as possible.

Try not to repeat students' answers. If students are to listen to one another and value one another's input, I cannot repeat or try to improve on what they say. If students realize that I will repeat or clarify what another student says, they no longer have a reason to listen. I must be patient and let students clarify their own thinking and encourage them to speak to their classmates, not just to me. All students can speak louder—I have heard them in the halls! Yet I must be careful not to embarrass someone with a quiet voice. Because students know that I never accept just one response, they think nothing of my asking another student to paraphrase the soft-spoken comments of a classmate.

Let students clarify their own thinking

"Is this the right answer?" Students frequently ask this question. My usual response to this question might be that "I'm not sure. Can you

6. Use hand signals often. Using hand signals—thumbs up or thumbs down (a horizontal thumb means "I'm not sure")—accomplishes two things. First, by requiring all students to respond with hand signals, I ensure that all students are on task. Second, by observing the responses, I can find out how many students are having difficulty or do not understand. Watching students' faces as they think about how to respond is very revealing.

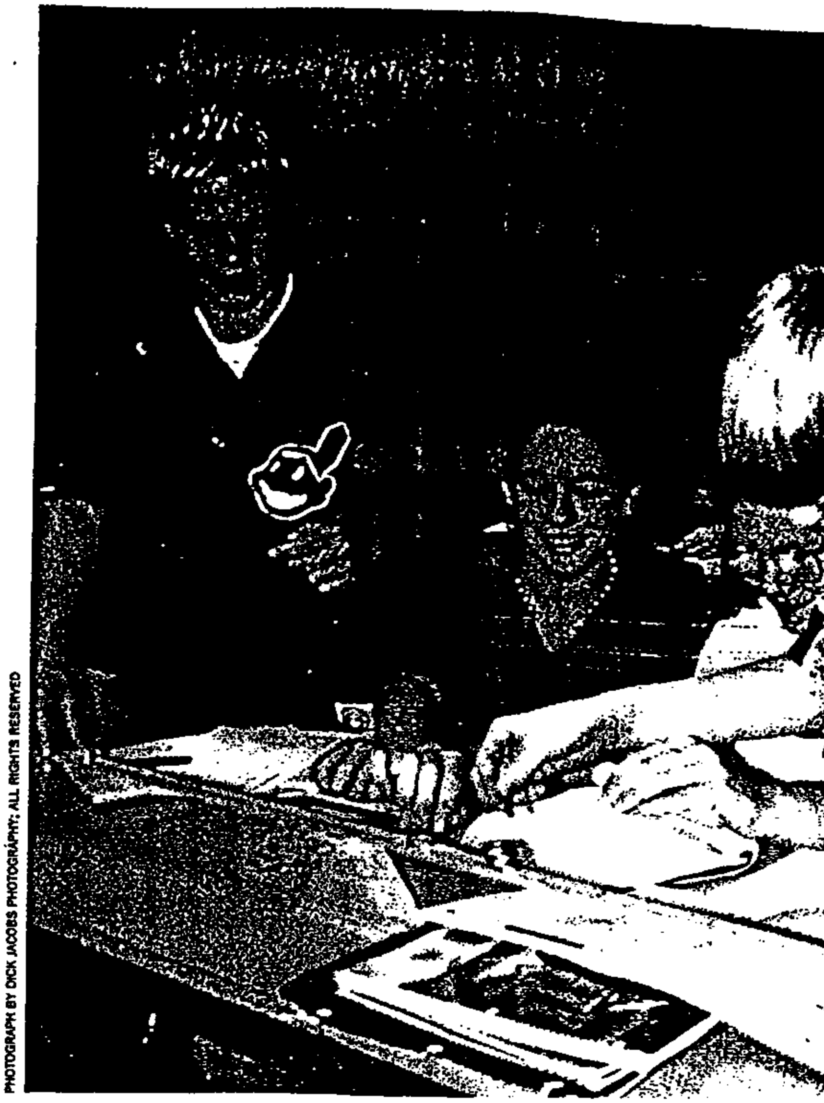
7. Never carry a pencil. If I carry a pencil with me or pick up a student's pencil, I am tempted to do the work for the student. Instead, I must take time to ask thought-provoking questions that will lead to understanding.

8. Avoid answering my own questions. Answering my own questions only confuses students because it requires them to guess which questions I really want them to think about, and I want them to think about all my questions. I also avoid rhetorical questions.

9. Ask questions of the whole group. As soon as I direct a question to an individual, I suggest to the rest of the students that they are no longer required to think.

10. Limit the use of group responses. Group responses lower the level of concern and allow some students to hide and not think about my questions.

11. Do not allow students to blurt out answers. A student's blurted out answer is a signal to the rest of the class to stop thinking. Students who develop this habit must realize that they are cheating other students of the right to think about the question.



Summary

LIKE MOST TEACHERS, I ENTERED THE TEACHING profession because I care about children. It is only natural for me to want them to be successful, but by merely telling them answers, doing things for them, or showing them shortcuts, I relieve students of their responsibilities and cheat them of the opportunity to make sense of the mathematics that they are learning. To help students engage in real learning, I must ask good questions, allow students to struggle, and place the responsibility for learning directly on their shoulders. I am convinced that children learn in more ways than I know how to teach. By listening to them, I not only give them the opportunity to develop deep understanding but also am able to develop true insights into what they know and how they think.

Making extensive changes in curriculum and instruction is a challenging process. Much can be learned about how children think and learn, from recent publications about learning styles, multiple intelligences, and brain research. Also, several reform curriculum projects funded by the National Science Foundation are now available from publishers. The Connected Mathematics Project, Mathematics in Context, and Math Scape, to name a few, artfully address issues of content and pedagogy.

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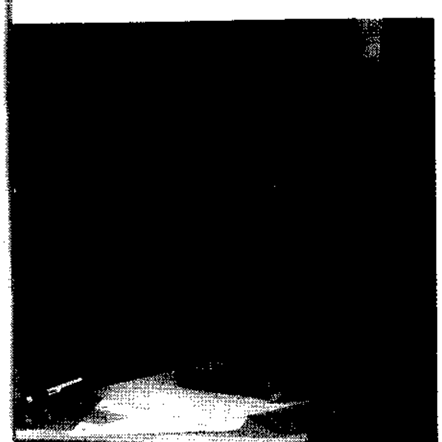
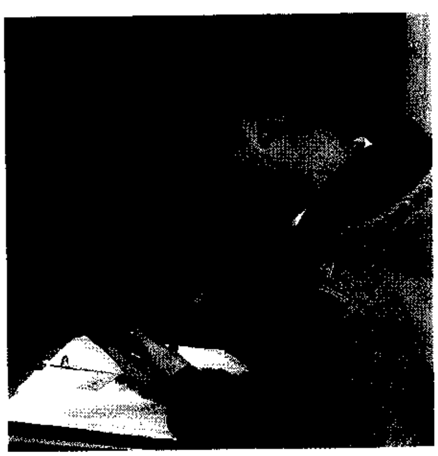
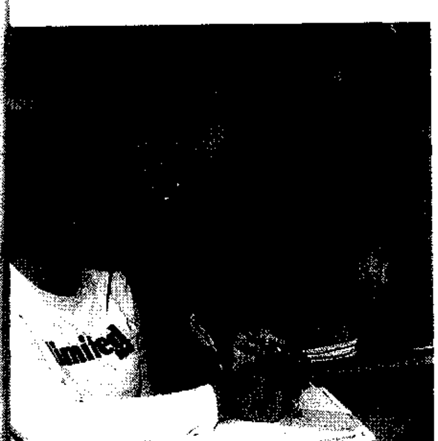
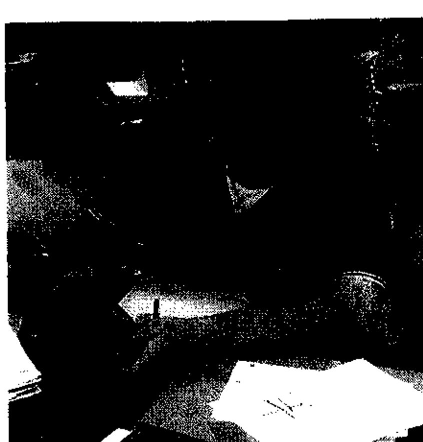
Shifting from Traditional to Nontraditional Teaching Practices Using Multiple Representations

Robin Rider









I have always been a strong proponent of using graphing calculator and computer technology in the classroom to enhance student understanding. I recognize that employing technology's ability to rapidly produce alternate representations, specifically graphs and tables, can strengthen students' awareness of symbolic expressions and equations as well. Although I realized this, my instruction could still have been classified as traditional in the sense that I taught students to move through the representations in one direction only. Starting with the symbolic representation, for example $y = 2x + 3$, I would have students create a table of values and then produce a graph. The graph was always the answer or final product. The table was used simply as a means of creating a graph. My assessments would be given in the same manner: The symbolic representation would appear as the assessment question, and then students would make a table and produce the graph for the answer. I believed that my instructional methods were aligned with the recommendations of the National Council of Teachers of Mathematics, which advocates a curriculum using multiple representations of mathematical concepts to promote connections among algebraic, tabular, and graphical representations in grades pre-K through 12 (NCTM 1989; NCTM 2000). Like many other teachers (Jacobs et.

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al 2006), I have come to realize that this belief was inconsistent with my actual classroom practice and with what NCTM recommends.

A SHIFT IN PEDAGOGY

When teaching solely in an equation-to-graph direction, the teacher makes the implicit assumption that when students are taught how to produce a graph, they will understand how it relates to the table and to the symbolic representation. For example, my students would be asked to rewrite the equation $2x + 3y = 6$ in slope-intercept form, identify the slope and y-intercept, and then graph the line, either by plotting points from a table of values or by using the slope and y-intercept. I made the implicit assumption that when I gave students a graph, they would be able to reverse the process to find the slope and the y-intercept from that graph and write the equation in slope-intercept form. I also assumed that given a table of values, they would be able to calculate slope and find the y-intercept from the table and write the equation.

When I examined how my students performed on state-mandated testing items involving tables and graphs, however, I realized they were missing an important piece of the representation puzzle. This missing connection involved the idea of reversibility (Krutetskii 1976) in which students not only can perform a procedure but also can reverse the process. Thus, they should be able to produce a table or a graph, but they should also be able to produce the other representational forms from a table or a graph. I never considered that with the prevalence of classroom graphing technologies quickly producing all the representations, my instruction could be designed to promote the use of the graphical and tabular representations simultaneously with the symbolic representation. This simultaneous coordination of representations would allow me to highlight the strengths and weaknesses that each representation affords and could be a powerful tool in developing algebraic and functional understanding (Kaput 1993).

Instructional emphasis on the manipulation of symbolic representation has been found to be one of the most significant influences on student preference of which representation is used to solve a problem (Yerushalmy and Schwartz 1993). This fact, combined with the realization that research has documented students' tendency to rely heavily on symbolic solution methods even when a graphical or tabular method may have been more efficient (Knuth 2000), made me question my own teaching. I started to consider how I could break this chain of student dependency on symbolic representations. Starting small, I endeavored to start each problem I did in class with a different representational form, even when introducing a concept for the first time. For example, when introducing solution of quadratic equations in standard form, I might start with a tabular representation and focus on where the y-coordinates were zero. Each time I did a problem, I would leave the representation I started with on the board or overhead and then solve the problem with each of the other representations. This allowed my students to work more fluently with all three representations, recognizing them as mathematically equivalent. For example, when finding the slope and y-intercept from a graph, a table, and an equation, my students recognize that "it's the same problem—just a different way to go about getting the answer."

Recognizing different representational forms of linear equations in two variables or the solutions of a quadratic equation may seem trivial to mathematicians, but for students in courses below college algebra, the ability to analyze a graph, a table, and an equation and find significant features within each, such as the slope and y-intercept, is very difficult. They tend to prefer a "cookbook" procedure for solving problems and do not want to be shown alternate representational forms. Comments such as "Just show us one way to do it and don't confuse us" can be heard on a daily basis. If those same reluctant students can begin to recognize invariance across representational forms and realize that each representational form is a viable way to solve a problem, then they have demonstrated significant gain in their understanding not only of the representations but also of the underlying concepts. This understanding gives them a choice of representations to use and lessens their reliance on procedural manipulation of the symbolic representation.

Experience strongly influenced what representation my students chose to use to solve a problem. "Because each representational format has varying limitations or strengths in different contexts, it is beneficial to have the choice of which representations to employ and the knowledge needed to make such a choice" (Lloyd and Wilson 1998, p. 253). For students to have this choice and this knowledge, they must have had experience with each type of representation.

A SHIFT IN ASSESSMENT

Even though I used all three representations when teaching the concept, my formal and informal assessments, such as tests, quizzes, homework assignments, and even questions asked in class, were always from a symbolic perspective. My students soon discovered that all the "extra stuff" I was doing in class with the graph and the table was not important. They learned to value the representation by which they were assessed, only using the alternate representations to check their answers to symbolic problems. The singular "aha" moment when I recognized my own bias revolutionized the way I teach and assess algebraic concepts.

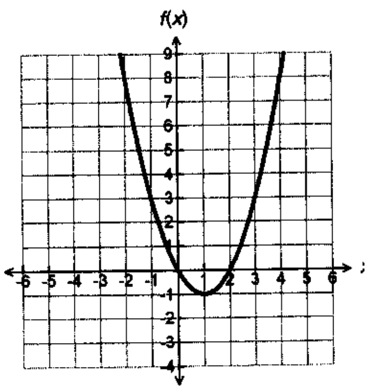
Given two functions: $f(x) = x^2 - 2x$
$g(x) = 2x + 1$

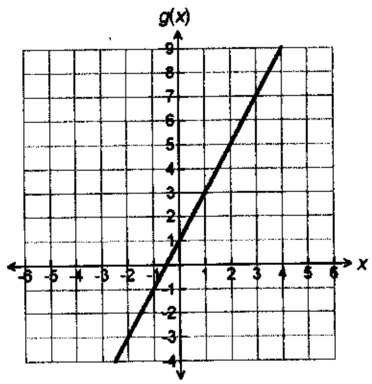
Answer the following:

1. Find $g(0)$.
2. Which is greater: $f(-1)$ or $g(3)$?
3. When is $f(x) = 0$?
4. Find the product of $f(4)$ and $g(-2)$.
5. Find $f(g(1))$.

Fig. 1 An example of typical questions on functions from a symbolic perspective

Given the functions:





Answer the following:

1. Find $g(0)$.
2. Which is greater: $f(-1)$ or $g(3)$?
3. When is $f(x) = 0$?
4. Find the product of $f(4)$ and $g(-2)$.
5. Find $f(g(1))$.

Fig. 2 The questions from **figure 1** from a graphical perspective

Given the functions:

x	$f(x)$	$g(x)$
–6	48	–11
–5	35	–9
–4	24	–7
–3	15	–5
–2	8	–3
–1	3	–1
0	0	1
1	–1	3
2	0	5
3	3	7
4	8	9
5	15	11
6	24	13

Answer the following:

1. Find $g(0)$.
2. Which is greater: $f(-1)$ or $g(3)$?
3. When is $f(x) = 0$?
4. Find the product of $f(4)$ and $g(-2)$.
5. Find $f(g(1))$.

Fig. 3 The same questions from a tabular perspective

I shifted my emphasis from symbolic representations to one that fostered the use of multiple representations by making small changes in the way I assessed. For example, when teaching function notation and how to evaluate a function for a certain value, the typical questions I asked of students were from a symbolic perspective (**fig. 1**). Even though my students were successful in evaluating functions from this perspective, I realized how little they knew when I asked the same questions from the graphical (**fig. 2**) and tabular (**fig. 3**) representations. Thus, I started asking questions, both informally in class and on formal assessments, from these alternate forms. I made it clear to my students that they might be assessed with any of the representational forms, reducing their desire to "only want to know one way to do it."

Modifying old assessments and incorporating questions asked from tabular and graphical forms was very difficult. In the beginning, I did not revise every exam I gave, but I started with concepts, such as functions, that can be easily tested from alternate representations. I took the symbolic questions I already had—for example, if $f(x) = 2x^2$, find $f(-2)$—and created a graph of $f(x) = 2x^2$. Then I asked students to find $f(-2)$ from the graph. Similarly, I might have revised a problem from a table. Taking assessments and topics I normally taught and shifting my thinking to incorporate alternate forms also shifted my dependency on teaching from symbolic representations.

A SHIFT IN BELIEFS

To limit the impact of my own representational preferences on my students' choice of representation, I considered how I could teach a concept using one of the other representational perspectives every time I planned my instruction for the concept. I made a conscious effort to start each new concept with a different representation and not to overemphasize any particular representation. I was not teaching anything "extra" but simply promoting fluency among the representational forms. "Fluency with multiple representations of mathematical relationships plays a significant role in the successful development of algebraic thinking" (Coulombe and Berenson 2001, p. 172).

What began as small changes in my teaching and

Find the equation of the line parallel to $2x + 3y = 6$ that passes through the point $(1, 4)$.

Fig. 4 Typical prompt to find the equation of a line parallel to a given line through a point not on the given line

assessing specific algebraic concepts became a huge shift in my beliefs as a teacher. Students quickly learned that they could be assessed with any of the representational forms and thus needed to be competent with each and could not rely on a particular representation. Incorporating multiple representations into the normal curriculum, highlighting each of the representations, and trying not to show any preference for one representation over another fostered representational competence in my students (Rider 2004).

A SHIFT IN FOCUS

The need for multiple representations is apparent when teaching concepts via mathematical modeling. Mathematical modeling involves the use of symbolic representation to model the behavior of real-world phenomena. In high school and beyond, students should be able to generate and use data to ascertain what type of function fits or models that data (NCTM 2000). This requires students to be able to look at data that may be in the form of a table, graph it to determine what type of function best models the data, and then determine symbolically the formula for that function. There is coordination among representations that does not follow the typical symbolic-to-graph direction. Perhaps this is why mathematical modeling can be so problematic for some students who have received traditional algebra instruction.

Multiple representations can also be very useful in the study of functions. The function concept is a fundamental idea in algebra (Selden and Selden 1992) and is uniquely suited to highlight strengths and weaknesses of the representations. Functions can be regarded as "a set of ordered pairs, a correspondence, a graph, a dependent variable, a formula, an action, a process, or an object (entity)" (Selden and Selden 1992, p. 4). To develop a meaningful understanding of the uses of function, students must have a strong grasp of all the ways in which it is possible to represent functions and should be able to move from one view to another. Although the problems highlighted in **figures 1–3** are mathematically equivalent and would be considered by most mathematics educators to be routine, when posed from the graphical and tabular perspectives these problems can be very problematic for pre–college algebra students who have been taught from a traditional, symbolic-to-graph direction.

Students who have traditionally been assessed on functional understanding with symbolic representations tend to respond to the questions in **figures 2** and **3** with statements such as "that [the table] is just a bunch of numbers" and "I have no idea what to do with them to find the answer. If you give me an equation, I can find the answer." These statements reflect a lack of understanding of the function concept. The students who made these statements were highly successful in their respective high school algebra courses, but their inability to work with graphical and tabular representations indicates their dependence on symbolic manipulation skills.

Find the equation of the line parallel to the graphed line that passes through the point shown on the graph.

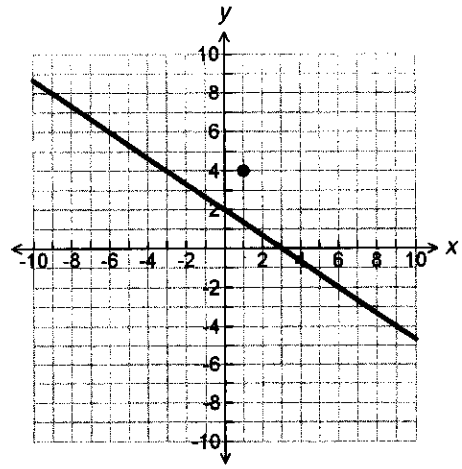


Fig. 5 The same question given a graphical representation

Linear equations are routinely taught in an equation-to-graph direction. When teaching parallel lines, for example, I used to start with the equation of a line and ask students to find the equation of the line parallel to the given line and through a given point (**fig. 4**). When I asked the same question of students but gave them a graph (**fig. 5**), I was dismayed to realize that they could not complete the same task. The same question asked from a tabular perspective (**fig. 6**) was even more problematic for students. Being able to solve the same problem from all three forms had seemed trivial to me; thus, I assumed that if students could do the symbolic problem, they would immediately make connections to the other two forms.

Concepts trivial to mathematicians are not necessarily trivial to pre–college mathematics students. Most mathematicians would consider the connections among the representations shown in **figures 4–6** to be trivial. To pre–college algebra students, however, the connections that start with either a table or a graph are not only nontrivial but in many cases nonexistent (see Knuth 2000; Rider 2004). The assumption that these connections will be apparent to students who learned in an equation-to-graph direction is typically

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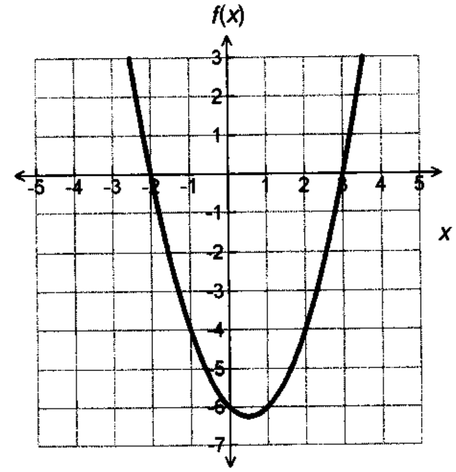
Find the equation of the line parallel to the line determined by the points given in the table and passing through (1, 4).

x	y
–12	10
–9	8
–6	6
–3	4
0	2
3	0
6	–2
9	–4
12	–6

Fig. 6 The question from **figure 4** given a tabular representation

Write $f(x)$ in factored form:
$f(x) = x^2 - x - 6$



x	y
–5	24
–4	14
–3	6
–2	0
–1	–4
0	–6
1	–6
2	–4
3	0
4	6
5	14

Fig. 7 Using multiple representations to highlight connections when teaching the factor theorem

false. Thus, I realized that to help my students become proficient with all the representational forms, I had to promote thoroughly each representation and assess students' ability to use the representations whether teaching with or without technology.

To foster representational competence in my students, I started with a concept that is part of the curriculum normally taught and tried to think of any way that I could view the concept from different representational forms. This was very foreign to the way I learned algebra. I found few resources to help me. (See the **appendix** for a brief resource list.) Much of what I changed in my teaching came about simply by contemplating how I could teach a concept differently. For example, a problematic area for many students is factoring polynomials. The factor theorem states that a polynomial $P(x)$ has a factor of $x - c$ if and only if $P(c) = 0$. When teaching the factor theorem, I would emphasize that $x - c$ is a factor of $P(x)$ if and only if c is a zero of $P(x)$ by writing out the theorem symbolically. I would then teach the students the different methods for factoring and have them practice, practice, practice. Viewing the factor theorem in a different light, I realized that by utilizing all the representations, I was able to highlight the zeros in the tabular representation and the x-intercepts in the graphical representation (**fig. 7**). This allowed students to observe the aspects of invariance across different representational forms, which made the factor theorem more accessible than it is when shown in symbolic form only.

This ability to observe aspects of invariance, such as noticing that the zeros in the tabular representation correspond to the x-intercepts in the graphical, had been coined by Slavit (1997) as a *property-oriented* view. If a student understood that solving a polynomial function equal to zero would produce the same solutions as the x-intercepts of the graph, he or she would be demonstrating a property-oriented view. This allowed students to make links between the two representations and the relevant properties of each needed to solve a factoring task.

A SHIFT IN TECHNOLOGY USE

A large body of research suggests that the use of graphing calculators as tools in mathematics classrooms has the potential to increase student understanding of functions, the connections among representations of functions, and the flexibility to move among the representations (see Hembree and Dessart 1992; Hollar and Norwood 1999; O'Callaghan 1998). Graphing calculators allow student and teacher to produce multiple representations of a function quickly and easily. This could be used to highlight features of each representation and to illuminate invariance among them, building students'

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flexibility to move between different representations. I realized that the methods in which I used the calculator in the classroom were key to these endeavors. Studies in which teachers continued to use traditional methodology when incorporating graphing calculators have shown that the calculator only provided more visual examples but did little to increase student understanding (Simmt 1997).

CONCLUSION

Skemp (1987) metaphorically describes instrumental and relational understanding with an example of different routes or paths in an unfamiliar city. Instrumental understanding, which is similar to procedural understanding, is likened to having one route that will get a person from home to office. The issue is, not whether the route is the best, but that a route exists: the person has a strategy for how to accomplish the commute. Relational understanding, which can be likened to conceptual understanding, is described as exploring a town and developing an internal cognitive map, which allows a person to take numerous routes, change direction, and get from any starting point to any finishing point. Traditional approaches to algebra and functions can be understood, using this same metaphor, as procedural or instrumental understanding: a set of rules and procedures to be mastered to get a person from point A to point B. Relational understanding, on the other hand, is developing a cognitive map, an understanding of algebra and functions that lets students draw upon numerous strategies and routes so that starting at any point, they can arrive at any other point. Teaching and assessing to promote representational fluency gives students the opportunity to construct their own cognitive maps.

To promote this fluency, teachers can start by taking concepts such as functions, equations of lines, and factoring and consider ways that the three representations can be introduced simultaneously, with no preference given to any one representation. This approach allows students to observe the strengths and weaknesses of each representation and learn which is best used as a tool in a particular situation. For students to learn to value each of the representations, they must also be assessed using all the representations. Teachers must think about how to ask questions in ways different from the typical symbolic-to-graph questions. From experience, I know that these small changes add up to enormous shifts in pedagogy, assessment, student achievement, and conceptual understanding.

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Appendix: Resources for Teachers

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Software

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ASSESSMENT ISSUES by Wendy Sanchez and Nicole Ice

Open-Ended Items Better Reveal Students' Mathematical Thinking

In this age of accountability, teachers need more—and more varied—data about their students' mathematical understanding than ever. One way to acquire these data is through the use of assessment items that are open-ended. Open-ended items "have more than one answer and/or can be solved in a variety of ways" (Moon and Schulman 1995, p. 25). In addition to producing an answer, students must also show their solution process and justify their answer. Open-ended items offer opportunities for students to demonstrate their mathematical thinking, reasoning processes, and problem-solving and communication skills. Because open-ended items invite a wider range of solutions and solution methods than more traditional assessment items, they are better at revealing students' understanding of mathematics.

Many of the mathematical questions that we ask students to answer allow them to reproduce memorized procedures without thinking about why the processes work and what the answer means. One of the authors recalls a test that she gave to second-year algebra students on solving systems of linear equations. Most teachers have given a similar test: first, students are asked to solve by graphing, then by substitution, then by elimination, and finally by any method that they choose. Once, after attending a workshop on assessment, the author added to her test the question "create a system of linear equations whose solution is (1,4)." Most students who were able to answer almost all the usual questions correctly were unsuccessful on the new question. These results made the author believe that her students had only a procedural understanding of systems of linear equations. They had memorized procedures that enabled them to produce correct answers, but they were not able to demonstrate an understanding of what the solution of a system of linear equations really means.

Although it is important to assess students' mastery of mathematical skills, it is also necessary to assess their conceptual understanding of mathematics. Often, just a little twist on the questions that we typically ask our students can yield the assessment opportunities we need. Consider the following original and revised items:

Original	Revised
1. Add $-2 + 4 + -7$.	1. Give three different integers whose sum is -5.
2. Reduce $\frac{58}{87}$ to lowest terms.	2. Write three different fractions that reduce to $\frac{2}{3}$. Show that your fractions reduce to $\frac{2}{3}$.
3. Find the area of a triangle whose base is 12 cm and whose height is 5 cm.	3. Draw a triangle whose area is 30 cm^2 and label its base and height. Explain how you know that the area is 30 cm^2.
4. Graph $y = -2x + 4$.	4. Draw the graph of a linear equation with negative slope that intersects the positive y-axis. Explain why your graph satisfies the given conditions.
5. What is the vertical asymptote of the graph of $y = \frac{x+3}{x-2}$?	5. Give the equation of a rational function whose graph has a vertical asymptote at $x = 2$. Explain why your equation satisfies the criteria.

Each revised item asks students to provide an example that satisfies certain criteria rather than asking them to carry out specific procedures.

It is important to recognize what students need to know and understand to respond to the revised item that they would not necessarily have to know and understand for the original item. In the original item 1, students have to know the rules for integer addition, whereas in the revised item 1 they also have to know what integers are and the meaning of the word *sum*. The original item 2 requires students to use a procedure to reduce a fraction; revised item 2 offers students the additional opportunity to demonstrate their understanding of equivalent fractions. In original item 3; students substitute values of b and h into the area formula. However, in the revision, students must also show that they understand that the base and height are perpendicular. In the original item 4, students routinely graph the point (0, 4), move "down two, over one" and graph a second point to determine the line. The revised item requires an understanding of the meaning of slope and y-intercept and their relationship to the coordinate plane. The original item 5 requires students to set the denominator equal to zero to get $x = 2$ as a vertical asymptote. In the revised item, they also have to know what a rational function is and what it means for that function to have a vertical asymptote.

Try using open-ended items as warm-ups, as homework, and—after students are used to them—on quizzes and tests. The students' responses will provide evidence about their mathematical thinking that is different from the kind of information given by the more traditional tasks that we ask of our students. It can only be helpful to have more and different information about our students' mathematical understanding.

Share your experiences with the column editors at nice@kennesaw.edu.

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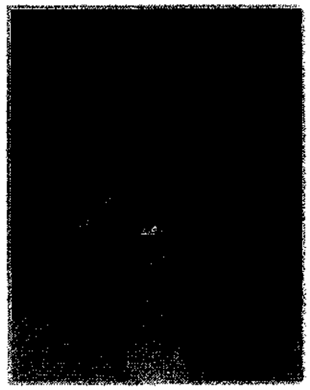
Ω

President's Message

A Journey in Algebraic Thinking

Cathy L. Seeley



The United States is one of the only countries in the world that is teaching courses with names like Algebra I or Algebra II. The rest of the world, including our colleagues in Canada, teach mathematics, not as separate courses, but as a continuous program from elementary through secondary school. In the United States, some schools offer an alternative, such as an integrated program that incorporates algebra as a strand blended with geometry and other advanced topics. Others continue to offer a course sequence that includes Algebra I, Geometry, and Algebra II. In an increasing number of states, the study of algebra in some form is required of all students for high school graduation. Regardless of whether a school's secondary curriculum includes a separate course in Algebra I or is more integrated, we can take concrete steps to ensure that students will flourish and succeed when they arrive at the formal study of algebra. A key to this success is the development of algebraic thinking as a cohesive thread in the mathematics curriculum from prekindergarten through high school.

Algebraic thinking includes recognizing and analyzing patterns, studying and representing relationships, making generalizations, and analyzing how things change. Of course, facility in using algebraic symbols is an integral part of becoming proficient in applying algebra to solve problems. But trying to understand abstract symbolism without a foundation in thinking algebraically is likely to lead to frustration and failure. Algebraic thinking can begin when students begin their study of mathematics.

At the earliest grades, young children work with patterns. At an early age, children have a natural love of mathematics, and their curiosity is a strong motivator as they try to describe and extend patterns of shapes, colors, sounds, and eventually letters and numbers. And at a young age, children can begin to make generalizations about patterns that seem to be the same or different. This kind of categorizing and generalizing is an important developmental step on the journey toward algebraic thinking.

Throughout the elementary grades, patterns are not only an object of study but a tool as well. As students develop their understanding of numbers, they can use patterns in arrays of dots or objects to help them recognize what 6 is or whether 2 is larger than 3. As they explore and understand addition, subtraction, multiplication, and division, they can look for patterns that help them learn procedures and facts. Patterns in rows and columns of objects help students get a sense of multiplication and see that facts make sense. Patterns within the multiplication table itself are interesting to children and help them both learn their facts and understand relationships among facts. The process of noticing and exploring patterns sets the stage for looking at more complex relationships, including proportionality, in later grades.

As students move into the middle grades, their mathematics experience can focus on connecting their work with numbers and operations to more symbolic work with equations and expressions. At this level, the focus of the mathematics program should be on proportionality, perhaps the most important connecting idea in the entire pre-K–12 mathematics curriculum. This concept should take students well beyond the study of ratios, proportions, and percent. A real understanding of proportionality allows students to connect their experience with numbers and operations to ideas that they have studied in geometry, measurement, and data analysis. They begin to get a sense of how two quantities can be related proportionally, as seen on maps, scale drawings, and similar figures, or in calculating sales tax or commissions.

A solid understanding of proportionality sets the stage for students to succeed in the more formal study of algebra. From this base, notions of linearity and linear functions emerge naturally. As students explore how to use linear functions to solve problems, the bigger world of functions that may not be linear begins to open for them. Looking at what is the same and what is different among functions lies at the heart of understanding algebraic skills and processes.

The journey doesn't end with a student's first formal study of algebra at high school. Continuous development of increasingly sophisticated algebraic reasoning can provide an avenue into the study of geometry and advanced mathematics. In the world outside school, these topics are not separated. When higher-level courses regularly incorporate opportunities to build on students' algebraic understanding, students are far more likely to succeed than if the courses present just one mathematical perspective.

The development of algebraic thinking is a process, not an event. It is something that can be part of a positive, motivating, enriching school mathematics experience. "Developing Algebraic Thinking: A Journey from Preschool to High School" is the Professional Development Focus of the Year for NCTM during the 2004–05 school year. Watch for opportunities to develop your own understanding of this important topic throughout the year, including as part of conferences, journals, publications, and the NCTM Web site.

For this month's questions, consider the following: How can we build algebraic thinking into the pre-K–12 curriculum at all levels? How should secondary school mathematics be organized to capitalize on the inclusion of algebraic thinking throughout the elementary and middle grades? What can NCTM do to support teachers in fostering the development of algebraic thinking?

Originally published in the *NCTM News Bulletin* (September 2004).

President's Message

Hard Arithmetic Is Not Deep Mathematics

Cathy L. Seeley

Helping all students develop a high level of mathematical proficiency is more important than ever before. Nearly every state and province has raised high school graduation requirements for students, and almost everyone agrees that we must raise our standards and expect more of our students. Attempts to define what it means to raise standards or increase expectations have led to interesting, and sometimes contentious, discussions at the state or provincial and local levels.

The message of NCTM's *Principles and Standards for School Mathematics* is clear. Students need a balanced mathematics program that allows them to be actively engaged in mathematics lessons so that they can develop deep understanding, mathematical thinking, and the ability to apply what they learn to solve problems. Computational proficiency is an important part of such a balanced program. However, computational proficiency is not the primary goal of effective mathematics programs. Instead, it is a tool used in the service of deeper mathematics.

The kind of mathematics that students need today—that adult citizens need—goes far beyond what once was sufficient. In the past, it might have been enough for a literate citizen to know how to read, write, and do basic measurement and arithmetic in everyday life. In the past, it might also have been enough for students who were going to college to master a set of algebraic tools that enabled them to take higher-level mathematics or science courses. But in today's world, there is rapid change, pervasive technology, and jobs that didn't exist five years ago. These all call for a much broader set of mathematical skills, including the ability to reason and apply mathematics to an ever-changing range of problems. And the reality of life today is that many more of our students are likely to participate in some kind of postsecondary education than ever before.

In this environment, how do we raise the bar on the mathematical proficiency that we expect of all students? And how likely is it that all students can achieve the goals that we set?

In response to the first question, we can raise the bar on mathematical proficiency by choosing fewer topics to focus on at each grade level and by teaching those topics in great depth. "Depth" means, for example, that students know a lot about multiplication before they deal with an algorithm for performing multiplication. "Depth" means that when fractions are introduced, we teach in such a way that students really know what fractions represent, in what kinds of situations they might be useful, how they compare to one another, how they relate to what students know about whole numbers, what it means when the numerator or denominator increases or decreases, and so on. "Depth" means that before students confront the rules for operating with fractions—such as going straight across, turning upside down, cross multiplying, etc.—we ensure that they know a lot about fractions and a lot about operations. "Depth" means that students go beyond "solving proportions" to recognize and utilize proportional relationships in ways that powerfully connect the ideas of prekindergarten–grade 12 mathematics. And "depth" means that students earning credit for a high school algebra course know how to solve equations and how to use algebraic tools and representations to solve many kinds of problems both within and outside of mathematics.

"Depth" does not mean making all students master arithmetic procedures earlier or with more digits. A school system whose standards include the mastery of fraction operations earlier than the standards of another system does not necessarily have a more rigorous curriculum. "Depth" does not mean narrowing our curriculum down to numbers and operations alone at the expense of measurement, geometry, and data analysis, where those numbers and operations are actually used. "Depth" does not necessarily mean more exercises. Focusing on more arithmetic procedures or more digits at the expense of deeper explorations and problem solving is not the same as raising our expectations for all students. And "depth" does not have to be painful or boring.

In visiting schools, I have found many wonderful examples where students are learning mathematics in depth. In these classrooms, mathematics is taught in greater depth and students are actively engaged, which opens the door for all students to master challenging mathematics. "Depth" is not the same as difficult arithmetic. "Depth" comes when students "get it." This means that students need to see the contexts in which mathematical ideas arise, need to wrestle with those ideas in problems that take some time to solve, and need opportunities to represent and communicate what they learn. The next President's Message will address the nature of student engagement in these classrooms and how we can ensure that students learn what is taught.

If we define our mathematics curriculum—the standards developed in our states and provinces—in ways that focus on students knowing and using mathematics and not just doing hard arithmetic, we can achieve this depth. And if we make some accompanying shifts in how we structure our classrooms, we can ensure that all students have an opportunity to reach the ambitious goals that we set.

Is your state or province or school system shifting its curriculum and standards toward deep mathematics rather than hard arithmetic? Do you believe that all students can achieve high standards? What will it take to make this happen? What will keep it from happening? Share your thoughts during my next online Presidential Chat, scheduled for 4:00 p.m. ET, Tuesday, October 26. Visit www.nctm.org at that time to join the discussion.

Also, in November be sure to read the President's Message about student engagement and join a related discussion online at 3:00 p.m. ET on Tuesday, November 16.

Originally published in the *NCTM News Bulletin* (October 2004).

NCTM Elaborates on Its Position Regarding Calculators

NCTM Positions and Position Statements define a particular problem, issue, or need and describe its relevance to mathematics education. Each statement defines the Council's position or answers a question central to the issue. The NCTM Board of Directors approves all positions and position statements.

The following position on the use of calculators in mathematics classrooms emphasizes a balance between the use of electronic aids and paper-and-pencil computation. NCTM's Board of Directors approved this position in May, and it is meant to clarify, rather than replace, NCTM's previously adopted policies on calculator use published in 2002 and 2003. NCTM positions and position statements are available online at www.nctm.org/about/position_statements.

Computation, Calculators, and Common Sense

A Position of the National Council of Teachers of Mathematics

Question

Is there a place for both computation and for calculators in the math classroom?

NCTM Position

School mathematics programs should provide students with a range of knowledge, skills, and tools. Students need an understanding of number and operations, including the use of computational procedures, estimation, mental mathematics, and the appropriate use of the calculator. A balanced mathematics program develops students' confidence and understanding of when and how to use these skills and tools. Students need to develop their basic mathematical understandings to solve problems both in and out of school.

Technology pervades the world outside school. There is no question that students will be expected to use calculators in other settings; this technology is now part of our culture. More important, when calculators are used effectively in the classroom, they can enhance students' understanding and use of numbers and operations. Teachers can capitalize on the appropriate use of this technology to expand students' mathematical understanding, not to replace it.

Written mathematical procedures—computational procedures in the elementary grades and more symbolic algebraic procedures as students move into the secondary level—continue to be an important focus of school math programs. All students should develop proficiency in performing efficient and accurate pencil-and-paper procedures. At the same time, students no longer have the same need to perform these procedures with large numbers or lengthy expressions that they might have had in the past without ready access to technology. Furthermore, computation should not exist in isolation. Measurement, geometry, and the analysis of data represent important mathematical content and provide useful contexts as students develop their numerical abilities.

Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math. These skills are essential for both understanding numbers and recognizing their usefulness outside school. Students should have a solid understanding of what addition, subtraction, multiplication, and division mean and how they work so that they can identify what operation(s) can help them solve a problem they encounter in math class, in another subject, or outside school. As they develop number sense, students acquire abilities to estimate and perform mental calculations quickly and proficiently. Students should become proficient at using mental math shortcuts, performing basic computations mentally, and generating reasonable estimates for situations involving size, distance, and magnitude.

A skillful teacher knows how to help students develop these abilities in a balanced program that focuses on mathematical understanding, proficiency, and thinking. The teacher should help students learn when to use a calculator and when not to, when to use pencil and paper, and when to do something in their heads. Students should become fluent in making decisions about which approach to use for different situations and proficient in using their chosen method to solve a wide range of problems.

Adopted July 2005

President's Message

Teaching to the Test

Cathy L. Seeley

ur tests are driving our teaching. This is the message from coast to coast as pressure mounts to produce results and meet the Adequate Yearly Progress requirements of the No Child Left Behind (NCLB) Act. Is this good or bad, and can a good mathematics program survive in this kind of environment?

Accountability is important in mathematics teaching. As professional mathematics educators, we must be able to demonstrate that our students are learning mathematics. Furthermore, the reporting of group data required by NCLB sheds light on gaps and problems within the mathematics program, including whether any group of students is achieving or not. Nevertheless, the kinds of tests that many states require, and the ways that many schools prepare their students for these tests, have serious limitations.

On the positive side, if a test assesses important mathematics in ways that require students to demonstrate mathematical thinking and proficiency, the test might effectively support a comprehensive mathematics program. For example, the state tests used in Connecticut and Washington call for students to complete a variety of mathematical exercises, including open-ended problems designed to require more complex thinking than what is called for in many state assessments. Students in a well-balanced mathematics program anchored in understanding, proficiency, problem solving, and mathematical thinking are likely to do well on these tests with or without special preparation strategies.

However, many state tests fall short of this ideal. Some are based solely on content that can be tested economically in a multiple-choice format, which often encourages students to try out all possible answers to a problem rather than actually solving it. Furthermore, although some state curriculum standards may include complex and high-level mathematical ideas, testing students' understanding of these ideas is not easy. This important content may get overlooked as teachers prepare students for items that are most likely to be included on the test. We must be cautious about the decisions that we make about students on the basis of such measures. No decision about a student's future should be based on any single measure, particularly a large-scale measure with inherent issues of context, bias, and intended purpose.

In too many schools, teachers are expected to "set aside" their mathematics program and instead prepare students for the state test. This may mean weeks or even months of missed instructional time. If preparing for the test means practicing a few items to get used to the format, it might serve students well. Too often, however, test preparation also includes learning tricks and tips that may or may not prove helpful on the test. For example, some schools use materials built on "clue words" for solving story problems or teach other tricks about what to do if presented with particular types of problems. Students memorize such phrases and words as *all together*, *more than*, and *total*, associating each with a particular operation. This type of practice falls apart on two levels. First, it misleads students. For any clue word or trick, most of us could create a test item for which the trick does not work. Second, the time that students spend memorizing tricks or words without understanding the related mathematics is precious time they lose from instruction that could support their mathematics learning. Students are better served by learning the concepts behind the numbers and operations so well that they carry mental pictures of what addition, subtraction, multiplication, or division mean. Recognizing a mathematical operation in the context of a problem and knowing how to perform the operation are far better preparation strategies than memorizing tricks or a list of words.

One other method of teaching to the test is periodic benchmark testing. Some school systems expect students to take tests throughout the year that are similar in format and content to the state accountability test. This can be an appropriate application of data-driven decision making. However, to be effective, any such strategy should be weighed according to cost and benefit. How much information is gained in a usable and timely manner for guiding and improving students' learning on a day-to-day basis? And what are the costs in instructional time and teacher time for planning, administering, interpreting and reporting results, and incorporating those results into the teaching process? These questions are essential to consider in any decision about testing and preparing for tests.

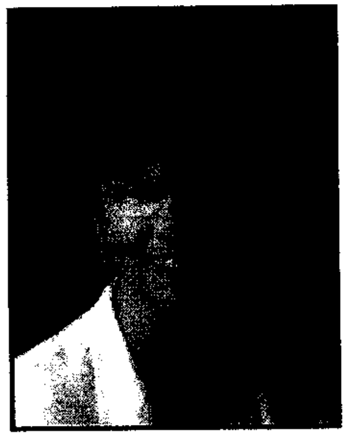
The best preparation for any test is teaching a good mathematics program well to every student. Even if the accountability test is a less-than-ideal measure, a strong mathematical foundation can prepare students to perform well. The reverse is not true, however. If we focus on test preparation at the expense of long-term learning, we may see short-term gains, but students are unlikely to be able to build on their learning from year to year. And some schools that devote excessive time to test preparation at the elementary grades may actually find, a few years later, that their middle school test scores have fallen. The bottom line is that professional mathematics educators need to be skeptical consumers of test-preparation programs and materials and knowledgeable judges of quality assessment practices that support students' learning. Most of all, professional mathematics educators need to be outspoken advocates for students, raising our voices when testing practices may not serve the best interests of students.

How can we balance teaching good mathematics and preparing for the state or provincial test? Are there effective test-preparation strategies that support student learning? Join me for an online chat about these and related questions on February 2 at 4:00 p.m. EST or submit your comments beforehand by visiting www.nctm.org/news/chat.htm.

President's Message

Do the Math in Your Head!

Cathy L. Seeley



What does it mean to know mathematics? This is a complex question, but there is strong agreement that facility with numbers and skill in problem solving play important roles. *Principles and Standards for School Mathematics* calls for students to be proficient with tools that include pencil and paper and technology, as well as mental techniques. I would like to make a case for raising the importance of *mental math* as a major component in students' tool kits of mathematical knowledge. Mental math is often associated with the ability to do computations quickly, but in its broadest sense, mental math also involves conceptual understanding and problem solving.

Mental Math Concepts

Understanding what numbers mean and what operations mean is the foundation for learning increasingly complex mathematics. Younger students should be able to recognize the number of objects represented in familiar patterns such as the five dots on the side of a die or eight objects arranged in two rows of four. For two- or even three-digit numbers, they might associate a numeral with a base-ten model that shows ones, tens, and hundreds. Students need mental pictures of a range of numbers like 10, 88, or 125. Carrying such mental pictures of the size and value of numbers prepares students for learning addition and multiplication facts and for solving simple problems involving computation. As students develop the mental ability to see numbers as being made up of other numbers (for example, seeing 125 as 100 + 20 + 5), their understanding of the number system expands.

Mental Computation

Ideally, students should have ready mental recall of their single-digit addition and multiplication facts. Ready knowledge of such facts for solving problems is an important component of mathematical knowledge. Beyond facts, I also believe that students should know how to multiply numbers mentally by 10, 100, or 1,000. Additionally, students should be able to come up with combinations that add up to 10 or 100. Students should know the pairs of whole numbers that add up to 10, realizing that, for instance, both 6 + 4 and 4 + 6 represent the same thing. When presented with a number like 37, students should be able to think of 63 as its "hundred partner." There are many other quick tips and mental shortcuts that can help students perform calculations mentally or that can aid them as they perform paper-and-pencil calculations.

Mental Problem Solving

Problem solving continues to be a high priority in school mathematics. Some argue that it is the most important mathematical goal for our students. Mental math provides both tools for solving problems and filters for evaluating answers. When a student has strong mental math skills, he or she can quickly test different approaches to a problem and determine whether the resulting path will lead toward a viable solution. Estimation skills require both a sense of number and facility with mental computation and can provide a ballpark answer to a problem before the student attempts to solve it. They also offer a comparison point by which to judge whether a result is reasonable for the given situation. Estimation is an important skill for inclusion in students' tool kits, whether they perform calculations with a pencil and paper or on a calculator.

Investing in Mental Math

Mental math proficiency represents one important dimension of mathematical knowledge. Not all individuals will develop rapid mental number skills to the same degree. Some will find their strength in mathematics through other avenues, such as visual or graphic representations or creativity in solving problems. But mental math has a clear place in school mathematics. It is an area where many parents and families feel comfortable offering support and assistance to their children.

Mental math need not depend on rote memorization. In fact, the development of mental models for numbers and operations is greatly facilitated by students engaging in purposeful experiences with concrete objects and number patterns. Teachers play a vital role in making sure that these experiences are connected in meaningful ways to the mathematics we ask students to learn.

In my observation, mental math does not receive the attention it deserves. Perhaps this is because the development of mental techniques is not always explicitly stated as an objective or state-level standard. Whatever the reason, the time has come to invest in helping students build the mental math skills in their tool kits as part of their comprehensive mathematical understanding. The payoff for this investment can be tremendous both in improving students' mathematical abilities and in giving a visible sign that we are committed to preparing students with the kind of mathematical proficiency that the public can readily appreciate.

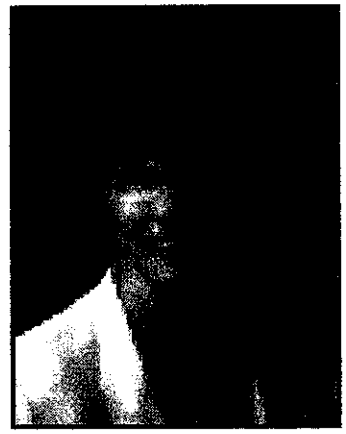
What are some of the most important mental math skills, tips, or shortcuts that students should know? What are some ways to help students develop their mental math facility? Join me for a chat about these questions and related issues on December 13 at 4:00 p.m. EST. Ω

Originally published in the *NCTM News Bulletin* (December 2005).

President's Message

Assessing to Learn and Learning to Assess

Cathy L. Seeley



With the public's attention focused on how well our schools are educating children in mathematics, students face assessment on a regular basis. At the classroom level, teachers use a variety of assessment measures and strategies to determine whether their students are learning the mathematics they are supposed to learn. Schools and school districts often administer tests across classrooms to determine whether students have reached a certain level of achievement or whether they are meeting particular benchmarks. Every state also administers large-scale accountability tests as required by law. Is all this testing too much, or does it help students learn? When we *assess to learn*, we seek information that allows teachers to find out what students know in order to improve their mathematics learning. This means *learning to assess* all the kinds of mathematical knowledge we identify as important. By assessing to learn and learning to assess, we can maximize the positive impact of assessment on students' learning without unnecessary negative consequences.

What Is Assessment For?

When a teacher wants to know whether a student or group of students is learning what is expected on a day-to-day basis, the teacher may use a variety of measures such as quizzes, interviews, projects, tests, or even purposeful conversation. If a test is used to determine the breadth of a student's mathematical knowledge and level of thinking, it is likely to include opportunities for the student to produce extended responses that demonstrate a thought process in addition to measuring a range of mathematical content and skills. To measure a program's effectiveness for a large number of students, a test needs to be efficient to administer and quick and economical to score. Each of these purposes calls for a different type of measure, with a specific format, scope, and context for administration.

Regardless of the purpose of an assessment, one factor is crucial to all assessments. They must be aligned with the particular mathematics that students are expected to learn. Ideally, this alignment will be evident in the content of the test or assessment and supported by the assessment format and manner in which the results are interpreted and used.

What Does Assessment Tell Us?

Teachers and students can benefit from assessment results that tell what a student knows and that identify a student's potential misunderstandings. The most useful assessment results for directly influencing students' learning are those that are immediate and specific. When we acknowledge what students are doing well and adjust or guide students as they first develop misunderstandings, they are far more likely to learn mathematics correctly and have that learning last. An effective teacher knows that it is important to measure students' understanding of mathematical concepts and ideas, evaluate their proficiency in skills, and give them the opportunity to apply what they have learned to a variety of situations beyond the immediate context in which the mathematics was learned. Results of large-scale accountability tests should be interpreted and used carefully only for the purposes for which they were intended. For making day-to-day decisions, the teacher is the best person to assess mathematics learning, and the classroom is the best context in which to do so.

Too Much Testing?

Ideally, assessment should be seamlessly woven into the fabric of teaching and learning, minimizing interruptions in instructional time and maximizing the immediate impact on students' learning. When assessment is part of the learning process, it does not need to sidetrack an effective mathematics program.

The bottom line is that the purpose of any mathematics assessment must be to improve students' learning. When an assessment measure is well aligned with—and integrated into—the system of mathematics teaching and learning, preparing students to perform well should involve little more than teaching the mathematics program well.

Learning to Assess

Developers of accountability assessments can improve assessments by incorporating problem solving, open-ended items, and problems that assess understanding as well as skills. As consumers of test data, teachers, supervisors, administrators, and families can learn what test data do and do not tell us. In our own classrooms, we can refine our skills so that we design assessment measures that clearly show what students know, with assessment integrated as part of the teaching and learning process. Throughout the 2005–06 school year, NCTM will provide focused resources on assessment to help us learn to assess and assess to learn. Look for the magnifying glass icon on the NCTM Web site and in NCTM publications for articles, ideas, and events that support this Professional Development Focus of the Year.

Meanwhile, let's start this school year by exchanging ideas about assessment. What ways have you found to determine how well your students are learning the mathematics expected of them on a day-to-day basis? Do you know of an effective and appropriate large-scale test to measure the mathematics students should know? Join me in a chat to talk about these and other related issues on Wednesday, September 21, at 4:00 p.m. ET. Submit comments and questions online at www.nctm.org/news/chat.htm.

Originally published in the *NCTM News Bulletin* (September 2005).

Board Explains NCTM Position on Large-Scale Testing

NCTM position statements define a particular problem, issue, or need and describe its relevance to mathematics education. Each statement defines the Council's position or answers a question central to the issue. The NCTM Board of Directors approves all position statements.

The following position statement describes how best to use tests and large-scale assessments to improve mathematics education. NCTM's Board of Directors approved this position in January. This and other position statements are available online at www.nctm.org/about/position_statements.

High-Stakes Tests

A Position of the National Council of Teachers of Mathematics

Question

What is the role of large-scale testing in making significant, high-stakes decisions about schools, students, and instruction?

NCTM Position

The National Council of Teachers of Mathematics recognizes the importance of measuring the learning of students and the effectiveness of instruction. Large-scale tests can and should be among several measures that are used in making significant decisions about students and instruction. However, such critical decisions about students and instruction must consider more than the results of any single test. We strongly support a balance of day-to-day classroom assessments, which help teachers improve instruction, and external tests that track progress and provide for national comparisons.

Large-scale tests are widely used in decisions related to promotion, graduation, admission to college, and school accreditation. Some view such high-stakes testing as a way to raise expectations and to hold students, teachers, and administrators accountable. Basing major decisions about students, teachers, schools, or instructional programs on a single test is inappropriate and inconsistent with what we know about learning and assessment. Tests, after all, are snapshots that capture one event in one context rather than a wide array of events in multiple contexts. The results of large-scale tests must be balanced against a broader sampling of student performance.

Valid, reliable large-scale assessments are useful and important tools for examining students' progress and making a variety of comparisons. However, because they may not measure the full range of important mathematics, they must be combined with a more complete sampling of student performance. This sampling might include classwork, tests, quizzes, observations, projects, and interviews. Such a collection of both informal and formal assessments can provide teachers and others with a more complete picture of student performance. By contrast, placing too much emphasis on a single test or on testing can undermine the quality of education and jeopardize equality of opportunity.

Given the pressures of high-stakes testing, teachers may commit too much instructional time to the mathematics that appears on tests. This mathematics is often limited to what can be readily tested in multiple-choice format. Furthermore, many large-scale tests focus disproportionately on simple mathematical outcomes. According to a recent study, "The most challenging standards and objectives are the ones that are undersampled or omitted entirely … [and those] that call for high-level reasoning are often omitted in favor of much simpler cognitive processes." (Achieve, 2003)

Assessment can and should be used to measure students' growth and inform instruction. Using information from a range of assessments, teachers can diagnose students' difficulties and strengths and modify instruction so that all students can increase their mathematics learning. Such a range of assessments should also be considered for high-stakes decisions about students and the effectiveness of instruction at the school and district level.

Adopted January 2006

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Claudia Wallis

The Myth About Homework

Think hours of slogging are helping your child make the grade? Think again

SACHEM WAS THE LAST STRAW. OR WAS IT *KIVA*? MY 12-YEAR-old daughter and I had been drilling social-studies key words for more than an hour. It was 11 p.m. Our entire evening had, as usual, consisted of homework and conversations (a.k.a. nagging) about homework. She was tired and fed up. I was tired and fed up. The words wouldn't stick. They meant nothing to her. They didn't mean much to me either. After all, when have I ever used *sachem* in a sentence—until just now?

As the summer winds down, I'm dreading scenes like that one from seventh grade. Already the carefree August nights have given way to meaningful conversations (a.k.a. nagging) about the summer reading that didn't get done. So what could be more welcome than two new books assailing this bane of modern family life: *The Homework Myth* (Da Capo Press; 243 pages), by Alfie Kohn, the prolific, perpetual critic of today's test-driven schools, and *The Case Against Homework* (Crown; 290 pages), a cri de coeur by two moms, lawyer Sara Bennett and journalist Nancy Kalish.

Both books cite studies, surveys, statistics, along with some hair-raising anecdotes, on how a rising tide of dull, useless assignments is oppressing families and making kids hate learning. A few highlights from the books and my own investigation:

■ According to a 2004 national survey of 2,900 American children conducted by the University of Michigan, the amount of time spent on homework is up 51% since 1981.

■ Most of that increase reflects bigger loads for little kids. An academic study found that whereas students ages 6 to 8 did an average of 52 min. of homework a week in 1981, they were toiling 128 min. weekly by 1997. And that's before No Child Left Behind kicked in. An admittedly less scientific poll of parents conducted this year for AOL and the Associated Press found that elementary school students were averaging 78 min. a night.

■ The onslaught comes despite the fact that an exhaustive review by the nation's top homework scholar, Duke University's Harris Cooper, concluded that homework does not measurably improve academic achievement for kids in grade school. That's right: all the sweat and tears do not make Johnny a better reader or mathematician.

■ Too much homework brings diminishing returns. Cooper's analysis of dozens of studies found that kids who do some homework in middle and high school score somewhat better on standardized tests, but doing more than 60 to 90 min. a night in middle school and more than 2 hr. in high school is associated with, gulp, lower scores.

■ Teachers in many of the nations that outperform the U.S. on student achievement tests—such as Japan, Denmark and the Czech Republic—tend to assign less homework than American teachers, but instructors in low-scoring countries like Greece, Thailand and Iran tend to pile it on.

Success on standardized tests is, of course, only one measure of learning—and only one purported goal of homework. Educators, including Cooper, tend to defend homework by saying it builds study habits, self-discipline and time-management skills. But there's also evidence that homework sours kids' attitudes toward school. "It's one thing to say we are wasting kids' time and straining parent-kid relationships," Kohn told me, "but what's unforgivable is if homework is damaging our kids' interest in learning, undermining their curiosity."

Kohn's solution is radical: he wants a no-homework policy to become the default, with exceptions for tasks like interviewing parents on family history, kitchen chemistry and family reading.

Or, in a nation in which 71% of mothers of kids under 18 are in the workforce, how about extending the school day or year beyond its agrarian-era calendar? Let students do more work at school and save evenings for family and serendipity.

> “HOMEWORK ... MAY BE THE SINGLE MOST RELIABLE EXTINGUISHER OF THE FLAME OF CURIOSITY.” —ALFIE KOHN

Bennett and Kalish have a more modest proposal. Parents should demand a sensible homework policy, perhaps one based on Cooper's rule of thumb: 10 min. a night per grade level. They offer lessons from their own battle to rein in the workload at their kids' private middle school in Brooklyn, N.Y. Among their victories: a nightly time limit, a policy of no homework over vacations, no more than two major tests a week, fewer weekend assignments and no Monday tests.

Why don't more parents in homework-heavy districts take such actions? So too many of us think it's just our child who is struggling, so who are we to lead a revolt? Yup, when it comes to the battle of homework mountain, we've got too many Indians and not enough sachems. ■

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Teaching with *Discovering Mathematics*

Teaching an investigation-based program, such as *Discovering Mathematics*, is quite different from teaching a more traditional program. When teaching a traditional program, you present new concepts, work through some examples, then assign problems that give students an opportunity to practice what you have just taught them. When teaching an investigation-based program, you pose a problem *without* telling students how to solve it. Students discover the new concepts themselves as they explore the problem while you facilitate, assess progress, and provide information on a need-to-know basis. After students have worked on the problem, you guide a class discussion that summarizes the key ideas. Some investigations in *Discovering Mathematics* vary from this model by providing an opportunity for students to explore a familiar concept more deeply, rather than to discover something new. In those lessons, students are exploring the world with mathematics that has been explained to them.

Transitioning from a traditional program to an investigation-based program requires you to change the way you think about yourself, your students, and the teaching process. For example, you will

- view yourself as a facilitator and guide, rather than as a disseminator of information.
- become comfortable with a more active and busier classroom.
- allow students to make mistakes as they try to figure things out.
- trust students to help and correct each other.
- be ready to take advantage of "teachable moments."
- be open to the possibility that the process will lead you to a deeper understanding of the mathematics.

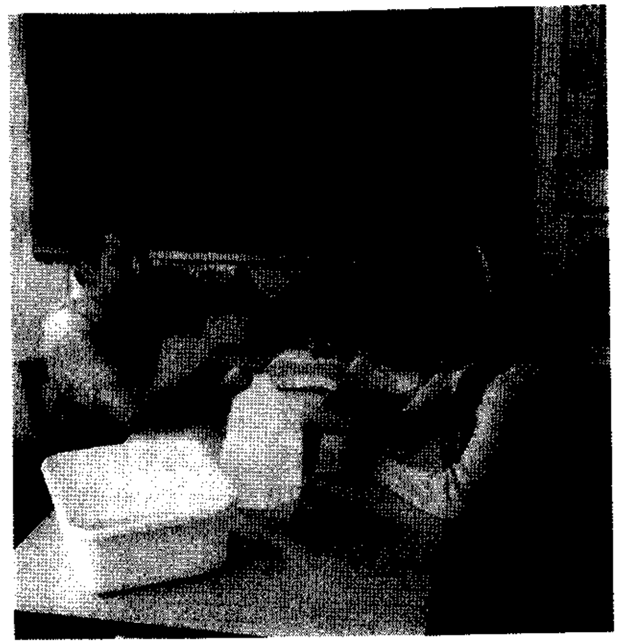
Once you become used to this new type of teaching, you may find it quite liberating. For example, you

- don't have to know all the answers; in fact, you can be open to the validity of students' answers even if they are different from your own.
- don't have to plan every minute of a class period; indeed, you can be quite flexible in response to student needs.
- can learn more about students' understanding without spending more time.
- can elicit and pursue questions that extend the mathematics of the lesson.
- can see students are learning as you watch them investigate and as you test them.
- can have more fun teaching.

Cooperative Learning

The *Discovering Mathematics* series engages students by giving them an opportunity to investigate mathematics in cooperative groups. As they work, group members will learn to plan together, brainstorm, determine and organize tasks, and communicate their individual and collective results.

Cooperative learning has many benefits.



- As students articulate ideas for other group members, they develop a deeper understanding of mathematical concepts and practice their oral communication skills.
- Students are exposed to more ideas for solving problems. When solving challenging problems, "two heads are better than one."
- Students learn and practice the essential life skill of working with others. In fact, developing cooperative skills is an important part of the curriculum.
- Students who have not been successful in mathematics, but who are comfortable in social situations, gain confidence in their mathematical abilities.
- Students learn to solve more complex problems than they otherwise could, without a group to contribute different areas of expertise.
- Group members can provide more immediate feedback to students than the teacher can.
- Students learn to respect and appreciate differences in ethnicity, physical and mental abilities, and learning styles.

FREQUENTLY ASKED QUESTIONS ABOUT COOPERATIVE LEARNING

Teachers who have not used cooperative groups in their classrooms often have questions and concerns.

Will the strongest or most motivated student(s) in each group do all the work and therefore most of the learning? If you leave groups to their own devices, one or two students may well do most of the work, especially if the entire group is receiving the same grade. These suggestions will help ensure that all students in the group do their share of the work.

- Assign tasks to group members then rotate those tasks.
- Require each student to produce a separate investigation report.
- Let groups know that the student from each group who will share the group's results with the class will be chosen at random when Sharing begins.
- Make group participation a part of each individual grade.

Is it fair to give everyone in a group the same grade? At the beginning of the school year, before all students are contributing to the work, you should not give group grades. Instead, you might assign grades in part according to the extent to which each student contributes to the group. Once you see group members are all participating, there may be an investigation, a project, or a group quiz for which you assign a group grade. It is never advisable to give a term grade to a group or to count group grades more than individual work on homework, quizzes, and tests.

Having students work in groups seems time-consuming. Will I be able to cover all the required course material? The *Discovering Mathematics* series is designed so that students will encounter all the major ideas of algebra and geometry through

investigations. As you watch students work in groups, you may sometimes get frustrated that it seems to take them more time to discover an idea for themselves than for you simply to tell it to them. When this happens, remind yourself of how many times you would need to repeat your explanation of an idea before students understood it deeply. Then note how well students remember it when they arrive at it through an investigation.

Won't my class become loud and out of control? While your classroom may be a little noisy, it is important to remember that a noisy class is not necessarily an out-of-control class. Guide students to understand what is productive, acceptable noise. You will probably have more control of your class if all your students are actively engaged in doing mathematics.

How can students teach each other? A great deal of learning mathematics is seeing patterns and making and testing conjectures. Students learn as they are exposed to different approaches and as they ask questions and talk with others about what they are thinking. They will not always discover a classic or efficient method for solving a problem, but they will understand the value of methods presented by others after they have struggled with their own methods. And when you introduce terms and help students formalize relationships after they have seen the patterns, they find it much easier to remember what you say. For example, the sample lesson from *Discovering Algebra* on page 17 shows students' development of the concept of slope before slope is defined.

Does having students work in cooperative groups require me to spend more time planning? Teaching in a cooperative-group setting requires about the same planning time as teaching in a traditional classroom setting. Some aspects of planning will take more time—you will need to think through the investigation and read the teacher's edition descriptions for guiding the investigation and helping students share their results. Some aspects of planning will take less time—you will not need to plan what you will be saying and doing every minute. (For more information on planning, see pages 14–16.)

FORMING GROUPS

Most investigations in *Discovering Mathematics* are appropriate for groups of three to four students. However, if you or your students have little experience working in groups, you might assign groups of four, but ask students to work in pairs for the first few investigations.

There are several strategies for assigning students to groups. At the beginning of the school year, when you don't know much about your students, it is probably easiest to assign groups randomly. As you get to know your students better, you can think about who might work well with whom. Mixing students of different ability levels can work. Or you might feel it is important to balance the groups for other factors, such as ethnicity, gender, or potential discipline problems.

Assigning students to groups might take an extra hour of planning each time you do it. Fortunately, you don't need to reorganize the groups very often. In fact, it is usually a good idea to keep students together long enough to get to know each other and learn to work together. Change the composition of groups occasionally, though, for variety of experience. You might reorganize groups at the beginning of each chapter or grading period or halfway through a grading period.

cooperative setting with desks. When using desks for group work, be sure the desktops are brought together to form as close to a single surface as possible, with no gaps between them. If possible, arrange the tables or desks so all students can see the front of the classroom by turning their heads rather than their entire bodies or chairs.

Discuss with your students why they are working in groups. Try to counter students' notions that learning is competitive, that a math course involves listening to information delivered by an all-knowing teacher, and that stronger students will suffer by working with weaker students. Emphasize that students will learn from each other when they see different ways of approaching a problem and that combining the different skills and ideas of group members can make solving a complex problem easier. Also, point out that good group skills are life skills that will help students succeed at work and at home.

Work with the class to develop specific guidelines for productive group work. Here are a few suggestions for group members.

- Speak in a low (two-foot) voice that can be heard easily by the group, but that does not disturb the rest of the class.
- Be considerate of others in your group.
- Listen without interrupting.
- Stay on task.
- Help others in your group.
- Be supportive.
- Ask questions if you don't understand.
- Criticize ideas, *not* people.
- Make sure everyone in the group understands the ideas well enough to present them to the class.

Give appropriate assignments. Good group assignments, such as the investigations in *Discovering Mathematics*, elicit many ideas or are large enough for group members to divide the tasks. The teacher's editions suggest ways to divide up steps of some investigations among groups. In *Discovering Geometry*, for instance, the *Teacher's Edition* suggests dividing theorem proofs among groups, then sharing, so all members of the class can add the theorem to the list of theorems they can use in further proofs.

Hold each group fully accountable. End each group session with group presentations to the class. You probably won't have time for all groups to present every time, but each group should be prepared.

Hold individuals responsible for group participation. Make part of the individual grade dependent on how the student contributes to the group. "Numbered heads together" is one method for ensuring that all group members understand the ideas from an investigation. Assign each group member a number from 1 through 4. Roll a die or spin a spinner to see which member of each group will present the group's ideas to the class. You can control the order in which the ideas are presented by choosing the order in which you call on groups.

Trust the group process. Give groups plenty of time to correct their mistakes. If a student asks you a question, turn it back to the group if possible. If some students are causing behavior problems, try to facilitate a group solution. For example,

VARIETIES OF GROUP ORGANIZATION

Groups should have a chance to work together in a variety of ways. The teacher's editions suggest ways of dividing tasks within or between cooperative learning groups.

Pair-share. A group of four splits into two pairs. Each pair does one part of the investigation or completes the investigation for one situation. The pair can divide up their work in various ways; perhaps one measures and the other keeps track of measurements, one reads the description of a situation while the other makes a diagram, one graphs an equation and the other checks. At the end of the investigation, the two pairs share and combine their work. This often works well when the group needs to gather a lot of data or explore several values of a parameter.

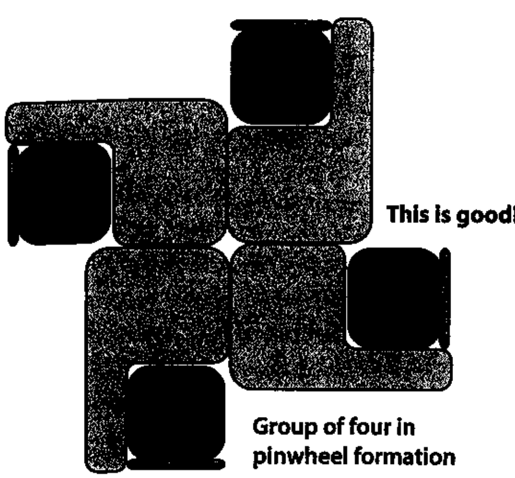
Jigsaw within a group. The work of the group is divided and assigned to different group members. They work individually on their part of the investigation, then they come together to share results. For example, each member might experiment with a different number, a different equation or family of functions, or a different geometric figure. After students finish their individual explorations, the group discusses and compares their findings and makes a conjecture or agrees upon a solution that they will share with the class.

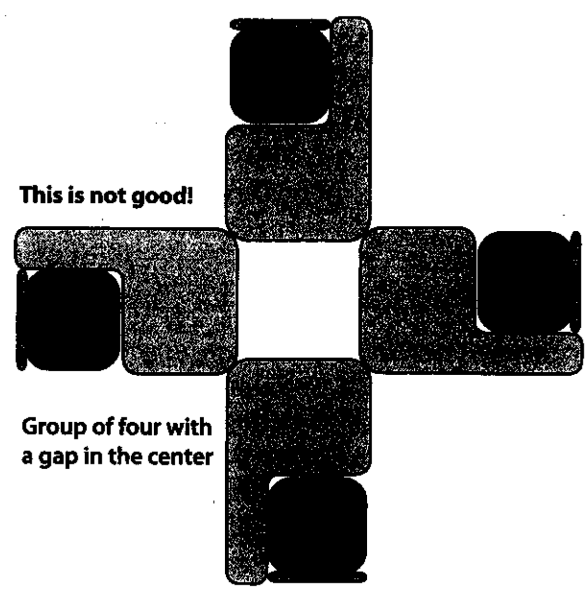
Jigsaw among groups. Sometimes an exploration or sequence of investigations can be broken up among groups so that each is working on a separate part. Your class might explore the sum of the angles of a polygon by having one group concentrate on quadrilaterals, another on pentagons, and so on. As groups share, your students look for patterns and find a formula that will apply to an n-sided polygon. Groups prove different theorems, then learn from each other's work when they share ideas.

FACILITATING EFFECTIVE GROUP BEHAVIOR

There are several things you can do to help group work go smoothly.

Arrange furniture appropriately. The type and arrangement of classroom furniture should be conducive to group work. Tables work best because they give your students a lot of work space, but it is also possible to have an effective





review the guidelines with the group or help students clarify their roles within the group. In extreme cases, you may need to call a student aside for a discussion of poor behavior. When you do so, also talk one-on-one with the other group members about expectations. It may help to point out that the groups will change soon, so students will have a chance to work with a different set of classmates.

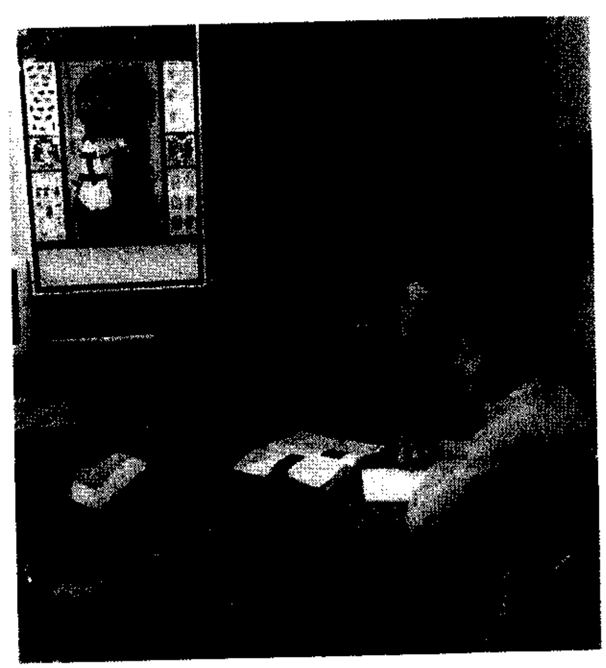
Your Role in an Investigation-Based Classroom

What do you do while your student groups investigate? After you have given students a few minutes to settle down and get started, begin to circulate among the groups, observing and encouraging as necessary.

- To facilitate groups, say little as you move around the room. Don't be too quick to jump in and correct a student who makes an error. If the other group members don't notice the error and if the error is a common one, make note of it and have a student present the error later so the entire class can learn from it. If students ask you a question, deflect it back to the group.
- Devise a plan to make use of students' ideas during Sharing time. Which group should present what? What questions should be raised? What points should be made? Which problem-solving strategies should be shared with the class and in what order should they be presented? Look for students who seem to understand various key steps particularly well. Be sensitive to the fact that students with poor calculation skills may have creative problem-solving ideas. Keep in mind the suggestions in the Sharing Ideas section of the teacher's notes.

You may find it helpful to think of yourself as an experienced lead investigator and manager, rather than as the person with all the answers. Here are some suggestions that might help you in this role.

Remove attention from yourself. As your students work and also when they present their results, consciously try to disappear from their awareness. Deflect questions back to students and wait for them to answer each other's questions. Move toward letting your students be more independent thinkers and learners. As students become better at investigating, they may not even notice you are visiting. Although you may feel unappreciated, independent student learning and critical thinking represent your ultimate success as a teacher!



Avoid micromanaging. Trust your students to think creatively and correct one another. Focus on helping them work more effectively. When students present ideas to the class, withhold your judgment, letting other students critique each presentation before you ask clarifying questions. After asking a question, give plenty of "wait time" for students to think and make sense of the ideas.

Do not assume you know the outcomes. Yes, you can solve the mathematics problem yourself, but you cannot really anticipate what sense students will make of it. Every mathematics problem can be approached validly in more than

one way. As you learn to resist your students' requests to tell them the "right" solution or the "best" solution and encourage them to think creatively, yet critically, they will begin to produce ideas you have not anticipated—ideas you can use to bring about deeper understanding. Ask students what led them to their approach, what reasoning process they followed. You want students to realize you are more interested in *their* answers than in *the* answer. Avoid comments and questions that suggest you know the right answer and are leading students to it.

Make constructive suggestions. Rather than giving omniscient hints, suggest good thinking strategies. Rather than telling students how to get unstuck, make suggestions such as "Sometimes you can understand the problem better if someone reads it aloud" or "One good problem-solving technique is to make a diagram." Be prepared to ask follow-up questions.

Help groups to stay on task. Go to any groups that aren't on task. If your simply joining them doesn't get them back on track, ask about their progress. If they think they've finished the task, look at their work, ask questions, make suggestions, and challenge them to extend their results. If students say they're stuck, ask one of them to describe what they've done, and ask others for their ideas about it.

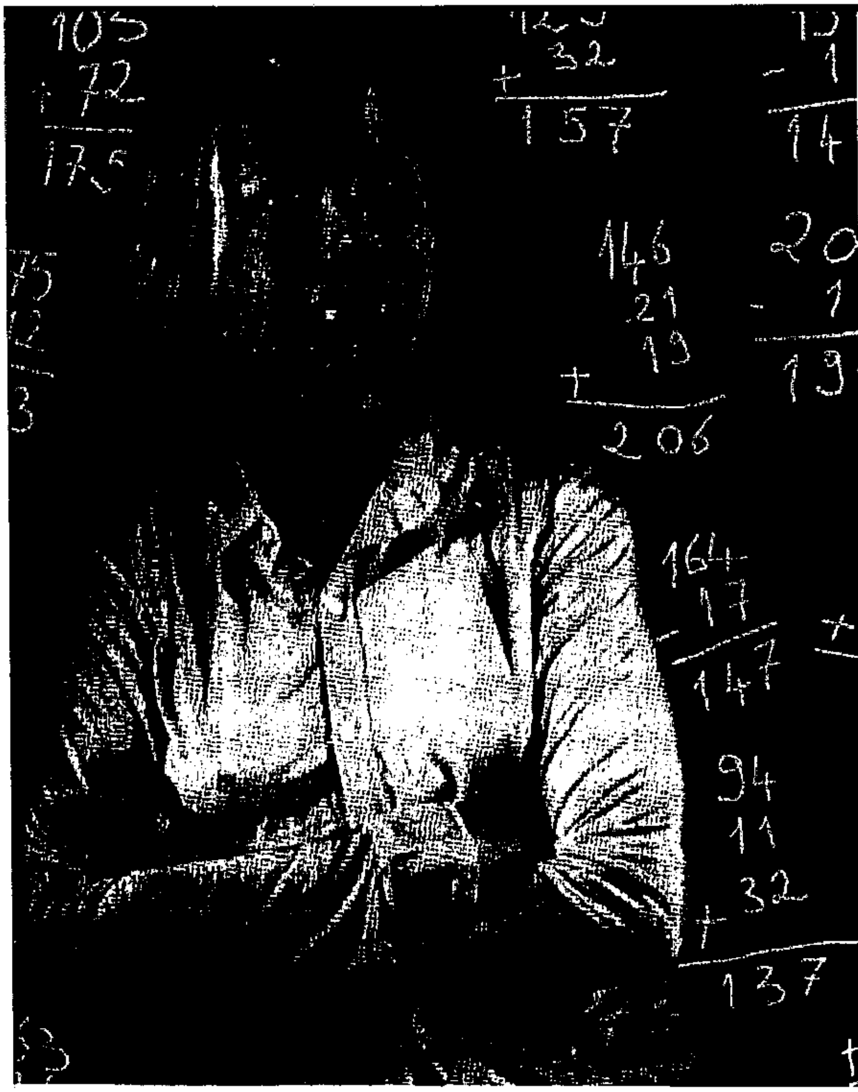
Give appropriate praise and encouragement. You cannot praise students for being "on the right track" when you don't know all the outcomes. You can, however, praise them for good thinking and cooperation. You can also encourage your students to learn from mistakes and persevere despite setbacks.

Take a long-term perspective. Note where the teacher's edition foreshadows future ideas, and think about the problem you have just assigned in that context. Be aware of the extent to which you must transcend the focus on solving the particular problem at hand and instead build your students' problem-solving and group skills so they can deal successfully with the mathematics that lies ahead. Focus on the process skills and habits of mind that will help students throughout their lives.

Manage students' interactions, not their thought processes. Control the working environment so students feel confident that they can contribute and so no student distracts others while they are working. Determine the order of presentations so that students get ideas from each other. Remember that students who may be labeled "underachievers" often achieve a great deal in this kind of environment.

Don't attempt to follow a "script." Following a script is impossible when you are acting as a manager rather than as a font of wisdom and knowledge. You must instead rely on your own skills as an investigator. Even if you have had very little experience investigating mathematics, your life experiences and knowledge of mathematics put you ahead of your students.

Work to dispel the notion that mathematics is a collection of facts and procedures. Help your students see that developing and applying appropriate problem-solving techniques and justifying ideas and results with sound reasoning are more important than getting the correct answer every time. Students should experience mathematics as a process of finding and connecting ideas. Let them know that the thinking and problem-solving skills they develop will serve them in all aspects of their life. They are learning far more from you than mathematical facts.



RESEARCH REPORT

What Does Good Math Instruction Look Like?

Nancy Protheroe

It involves good teachers, an effective math environment, and a curriculum that is more than a mile wide and an inch deep.

IN BRIEF

Our research-based knowledge about good math instruction, although not as extensive as that focused on reading instruction, has increased in recent years. It now provides a solid base of information for educators to use as they identify mathematics skills students need to develop, as well as teaching strategies and instructional approaches that best support the development of these skills.

What Gets Taught

When considering content knowledge and skills, it is obvious that schools must look first at the state standards that students are expected to master. However, research comparing math instruction in the U.S. and other countries has pointed to an underlying problem with many of our standards-based systems. Typically, these systems address too many standards for each grade level—encouraging the development of a curriculum that has been characterized as "a mile wide and an inch deep."

In contrast, the National Council of Teachers of Mathematics (NCTM) has developed "Curriculum Focal Points," a report that identifies three broad—but critical—mathematical concepts that should be addressed in each grade (see Web resources section on how to access more information about the report). For example, NCTM identifies these math focal points for second grade:

- *Number and Operations.* Developing an understanding of the base-10 numeration system and place-value concepts;
- *Number and Operations and Algebra.* Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction; and
- *Measurement.* Developing an understanding of linear measurement and facility in measuring lengths.

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SADIK DEMIROZ/SUPERSTOCK

NCTM suggests that state boards of education and other groups developing standards use the focal points as a "clear organizational model for establishing a mathematics curriculum from pre-kindergarten through grade 8" (NCTM 2006). At the local district and school levels, teacher conversations and staff development could be organized around the focal points. Encouraging teachers from several grades to participate in such a setting ensures discussions are also focused on the linkage of math instruction from grade to grade.

How It Gets Taught

During the reading wars between proponents of a whole language approach and those favoring skills-based instruction, educators found that a careful and intensive review of research revealed the importance of using a combination of both approaches. Similarly, there are at least two camps prominent in the discussion of how math should be taught. The two teaching approaches have clear differences. In skills-based instruction, teachers focus on developing computational skills and recall of facts. In the second approach, teachers encourage students to explain how they arrived at a solution and to consider more than one way of solving a problem.

Ideally, teachers should strive for a balance between the two approaches. Doug Grouws (2004), recently honored for his long-time contributions to mathematics education with the NCTM Lifetime Achievement Award, talks about this:

> *Research suggests it is not necessary for teachers to focus first on skill development and then move on to problem-solving. Both can be done together. Skills can be developed on an as-needed basis, or their development can be supplemented through the use of technology. In fact, there is evidence that if students are initially drilled too much on isolated skills, they have a harder time making sense of them later.*

"... Research comparing math instruction in the U.S. and other countries has pointed to an underlying problem with many of our standards-based systems."

What should effective mathematics instruction look like? Shellard and Moyer (2002) identify three critical components: "Teaching for conceptual understanding, developing children's procedural literacy, and promoting strategic competence through meaningful problem-solving investigations."

Also, topics should be presented in a sequence and manner appropriate for the developmental level of the students (Reys *et al.* 1999). Although the rate at which children develop mathematically varies from child to child, NCTM (2001) has developed a general timeline for students' mathematical skills development and instruction identified as appropriate for each level. According to this timeline:

- From pre-kindergarten through second grade, children develop a mathematical foundation by building beliefs about what mathematics is and what it means to understand and "do" mathematics. Instruction should be provided that helps them understand patterns and measurement and develop a solid understanding of the numeration system.
- Building on the inquisitive nature of children in grades 3 through 5, students should be encouraged to develop and investigate solutions to everyday problems. Instruction should focus on the relationship between such processes as addition and multiplication, and subtraction and division. Students should be introduced to multiplicative reasoning, equivalence, and a variety of methods for computation. Instruction at this level should also focus on developing children's interest in mathematics.
- Students in grades 6 through 8 are forming conclusions about their mathematical abilities, interest, and motivation that will influence how they approach mathematics in later years. Instruction at this level should build on their emerging capabilities to think hypothetically, comprehend cause and effect, and reason in both concrete and abstract terms. Algebra and geometry form a large part of the recommended curriculum during these years.

An important key to developmentally appropriate mathematics instruction, at any age or grade level, is achieving balance between teaching for conceptual understanding and teaching for procedural fluency. When students learn procedures without meaning, they are only memorizing discrete pieces of information that are difficult for them to remember. Students should develop an understanding of the concepts they are studying before they apply these ideas to procedural strategies.

Good Teaching Is Key

Of course, effective mathematics instruction begins with effective teaching. No lesson, no matter how well planned, can be successful if the elements of effective teaching are not in place. Grouws (2004) discusses the instructional practices that research has shown to have a positive impact on student learning and then mentions the role of the teacher:

> *The quality of the implementation of a teaching practice also greatly influences its impact on student learning. The value of using manipulative materials to investigate a concept, for example, depends not only on whether manipulatives are used, but also on how they are used with the students. Similarly, small-group instruction will benefit students only if the teacher knows when and how to use this teaching practice.*

In addition, to effectively develop students' mathematical skills, teachers must be effective overall. They must exhibit good classroom management skills, especially in classrooms using differentiated instruction; actively engage their students; and make efficient use of instructional time. A mathematics lesson cannot succeed if the other elements of teaching—classroom management, a logical progression of lessons, an effective use of assessment, and time management—are not in place.

An Effective Mathematics Environment

There are some specific teacher behaviors that "matter" in the teaching of mathematics. In effective classrooms, teachers:

- ***Demonstrate acceptance of students' divergent ideas.*** They challenge students to think more deeply about the problems they are solving and ask them to explain the solutions. Such an approach also helps students develop confidence in their own abilities to do mathematics and gain an even firmer grasp of key concepts and processes.
- ***Influence learning by posing challenging and interesting questions.*** Teachers should present questions that stimulate students' curiosity and encourage them to investigate further. The questions should encourage students to rely on themselves and their peers for ideas about mathematics and problem-solving.
- ***Project a positive attitude about mathematics and about students' ability to "do" mathematics.*** This includes demonstrating enthusiasm for the content as well as a belief that all students are capable of learning the material, with lessons designed to encourage curiosity, interest, and skill-building.

Certain instructional characteristics also are associated with effective mathematics instruction. By integrating the following approaches into classroom instruction, teachers can promote both student learning and motivation:

- ***Students are actively engaged in doing mathematics.*** They should not be sitting back watching others students solve problems.
- ***Students are solving challenging problems.*** Mathematics is a stimulating and interesting field generating new knowledge every day, and students should be exposed to this excitement and challenge, using real-world examples when possible.
- ***Interdisciplinary connections and examples are used to teach mathematics.*** For example, using literature as a springboard for mathematical investigation is a useful way to introduce authentic problem-solving situations that may have "messy" results. This engages students in connecting the language of mathematical ideas with numerical representations and

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develops important skills that support students' abilities to solve word problems.

- ***Students are sharing their mathematical ideas while working in pairs and groups.*** Research shows that students who work in groups on problems, assignments, and other mathematical investigations display increased achievement. Such opportunities appeal to the social nature of most children, while thinking through problems collaboratively makes it less likely that a student will get caught in a procedural dead end.
- ***Students are provided with a variety of opportunities to communicate mathematically.*** During a lesson, students should have many opportunities to communicate their ideas. They may draw a picture to represent their ideas or write them in mathematics journals. Whole-class discussions should provide opportunities to hear about and perhaps challenge other students' ideas in an environment of respect and understanding.
- ***Students are using manipulatives and other tools.*** The long-term use of mathematics manipulatives is positively related to student achievement and attitudes about mathematics. It is not enough, however, to simply provide students with manipulatives; they must be taught how to use these materials. Several steps can be taken to ensure students benefit from a lesson involving manipulatives. First, the teacher should use manipulatives that support the lesson's objectives. Next, before allowing students to handle the materials, the teacher should demonstrate how to use the manipulatives and the procedures for handling them. And finally, the lesson design should encourage the active participation of all students (Ross and Kurtz 1993).

An Ongoing Dialogue

Conversations about math instruction will continue. Parents, educators, policymakers, and future employers are all concerned about what—and how well—our students learn mathematics.

In recognition of the importance of the topic, a National Mathematics Advisory Panel was recently established within the U.S. Department of Education. Its charge is "to foster greater knowledge of and improved performance in mathematics among American students … with respect to the conduct, evaluation, and effective use of the results of research relating to proven-effective and evidence-based mathematics instruction" (Bush 2006). Members of the panel have been assigned to four task forces focused on critical areas of mathematics instruction: learning processes, conceptual knowledge and skills, instructional practices and materials, and teachers and teacher education (National Mathematics Advisory Panel 2007).

For teachers and others responsible for ensuring our students receive the best possible mathematics education, ongoing efforts to stay informed about current research findings will be as important as the frameworks provided by state standards. As principal, your role in ensuring that happens in your school is critical. P

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Shellard, E. and P. S. Moyer. *What Principals Need to Know about Teaching Math.* Alexandria, Va.: National Association of Elementary School Principals and Educational Research Service, 2002.

WEB RESOURCES

Information provided on the NCTM Web site includes the council's focal points by grade as well as questions and answers about the focal points.
www.nctm.org/focalpoints.aspx?ekmensel=c580fa7b_10_48_btnlink

"Mathematical Understanding: An Introduction," a chapter from the book *How Students Learn: History, Mathematics, and Science in the Classroom*, can be accessed at the National Academies Press Web site.
www.nap.edu/catalog.php?record_id=10126#toc

The K-12 Mathematics Curriculum Center, funded by the National Science Foundation, provides information on textbook adoption and a downloadable guide to several mathematics curriculum programs.
www2.edc.org/mcc/default.asp

TITLE: What Does Good Math Instruction Look Like?
SOURCE: Principal 87 no1 S/O 2007

TEACHER REFLECTION TOOLS

UNDERSTANDING, INVENTION, SENSEMAKING

Does classroom activity center on mathematical understanding, invention, and sensemaking by all students?

1) Do student explanations and justifications emphasize the *meanings* of ideas and *how* and *why* the students' methods do or don't work?
2) Do students determine the correctness/sensibility of an idea or solution based on the mathematical reasoning presented?
3) Are student conjectures, generalizations, mathematical justifications, "what-if" questions, and invented procedures the norm?
4) Do students approach problems and ideas in a variety of ways and using a variety of representations (visual, verbal, numerical, algebraic, graphical, and/or everyday life)?
5) Do all students use models, manipulatives, and other mathematical tools to make sense of ideas, solve problems, and invent procedures (or is use of such tools limited to teacher or student demonstrations)?
6) Do students use *genuine questions, statements*, and *actions* that show *genuine interest* in others' thinking about mathematics (or do actions/interactions center on getting others to think in certain ways)?
7) Do students listen intently and actively and ask for clarification when they don't understand someone's methods or reasoning?
8) Is *private think time* honored and encouraged by all?
9) Do students celebrate their mathematical AHA!s and honor the disequilibrium that may precede such moments?

MATHEMATICAL CULTURE

Is the classroom culture such that inquiry, wrong answers, personal challenge, collaboration, and disequilibrium provide opportunities for new learning by all students?

1) Do students explain, question, and debate their mathematical conjectures, reasoning, representations, generalizations, and justifications about ideas and problems?
2) Do students rely on their own thinking and the mathematical logic and structure of ideas to judge the correctness/usefulness of the ideas (or do they defer to others for authority based on status)?
3) Are mathematical contradictions a regular source of reflection and discourse?
4) Are *we wonder... we conjecture... we reason...* and *we can prove...* statements about relevant mathematical ideas central to collaborative investigations?
5) Are wrong answers viewed as worthwhile – as sites for learning?
6) Do students willingly report where their thinking is "stuck" and control the level of input from others regarding how to get "unstuck?"
7) Do students have autonomy in choosing and sharing their methods of solving problems?
8) Do students honor their own and each other's right and capacity to solve problems?
9) Are students equitable in their spoken and unspoken messages about all students' mathematical potential?

WORTHWHILE MATHEMATICAL TASKS

Is the lesson/task mathematically worthwhile for all students?

1) Do student actions and interactions focus on understanding and invention of important and relevant mathematics concepts, processes, and relationships?
2) Do student solutions involve thinking that is complex and nonalgorithmic (i.e., there is no prescribed solution path)?
3) Do students search for multiple solution strategies and recognize task constraints that may limit solution possibilities?
4) Do all students reflect intensively and critically on their own and each others' processes and reasoning?
5) Do all students have access and experience challenge and mathematical insight?
6) Do students experience mathematically productive disequilibrium?
7) Does the lesson/task suggest a mathematical vision and purpose beyond the immediate? That is, is there evidence of a meaningful and coherent *mathematical storyline* in development?
   a) Do students draw upon/link underlying math concepts and prior understandings to work through the task?
   b) Do students construct strategies and conceptions that are fundamental to understanding more complex math ideas?
   c) Do students generate and explore conjectures, generalizations about the mathematics fundamental to the task/lesson and where it is headed?
   d) Do students connect the mathematics ideas to relevant everyday life and/or other mathematics contexts?
8) Do students use multiple representations (visual, verbal, numerical, algebraic, graphical, and/or everyday life) graphical, and a variety of manipulatives and/or other mathematical tools as supports for sensemaking, problem solving, connecting concepts, and inventing strategies and algorithms?

MATHEMATICAL KNOWLEDGE FOR TEACHING

Does my knowledge of the mathematics content I teach, how students learn that content, and the trajectory of the mathematics enable me to support important, long-lasting student learning?

1) Do I pose questions and tasks that foster –
   a) student conjectures, connections, representations, justifications, and generalizations involving core mathematical ideas?
   b) mathematical contradictions and productive disequilibrium?
   c) student thinking and discourse about the *meanings* of underlying mathematical ideas (i.e., beyond explanations of procedures for solving a problem)?
   d) student consideration of ideas that are relevant to the mathematical intent of the lesson but that may not otherwise surface?
   e) access, engagement, and challenge for all students?
   f) multiple representations and explanations of concepts?
   g) connections among mathematical ideas, between math and other subjects, and between math and everyday life?

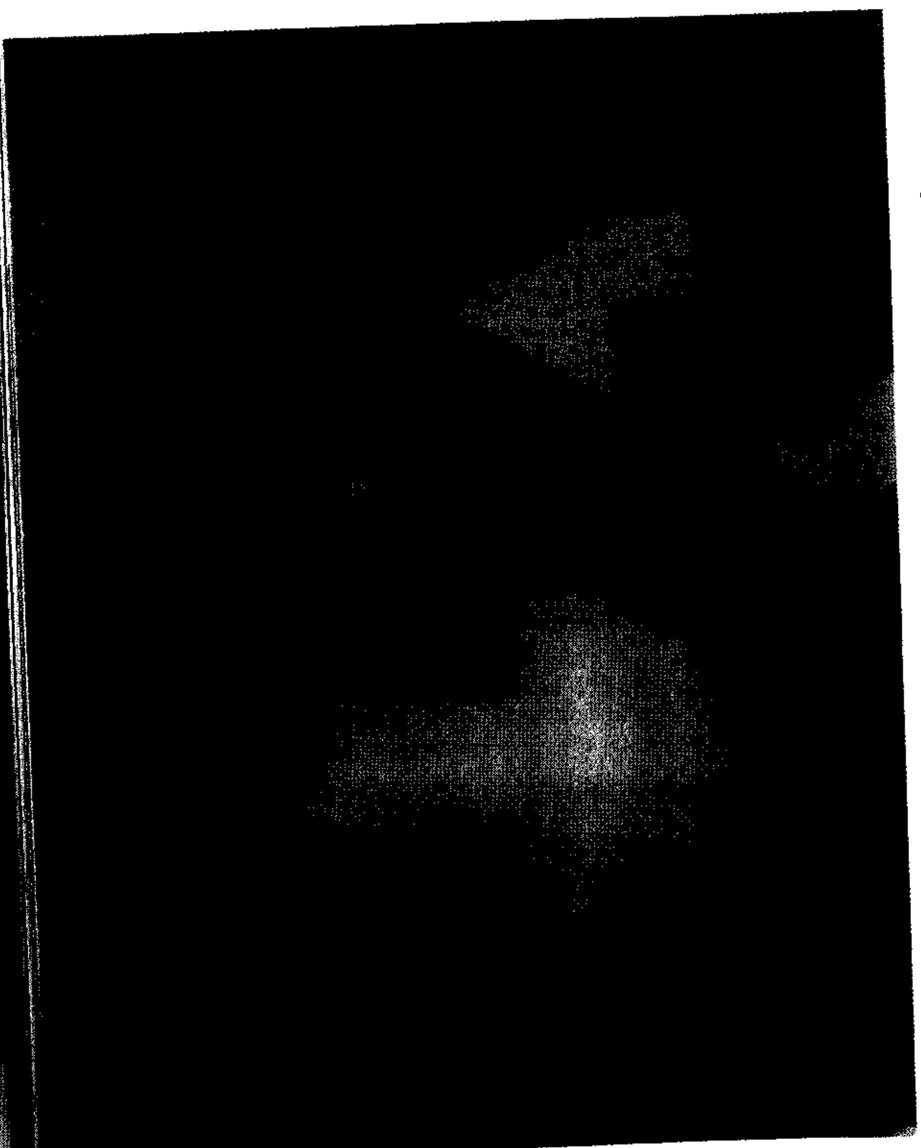
2) Do I listen intently to my students' thinking and respond according to its mathematical implications, i.e., according to –
   a) the alignment of students' thinking with research-based information on the development of students' mathematical thinking?
   b) the mathematical validity of students' thinking?
   c) the conceptual roots of the mathematical ideas at hand and/or the mathematical direction in which we are headed?

3) Do I ground my decisions about whether/how to pursue a student-generated question or idea according to its potential to be mathematically fruitful and its relevance to our mathematical agenda?

4) Do I openly learn and grow with my students?

The Inverse Name Game

Christine C. Benson and Margaret Buerman



The names we give to specific cases of a mathematical concept sometimes get in the way of understanding the concept itself. Over the years, as we have taught methods courses for preservice mathematics teachers, we have become increasingly concerned that the general concept of *inverse* has gotten lost. Observations of high school classrooms and personal tutoring sessions with high school students have further justified our concerns. As teachers, we may address these concerns and build conceptual understanding through discourse with students and through our instruction by drawing connections between different representations of inverses.

An example that illustrates our concerns may be found in the responses to a test item borrowed from Teresa Barry, a teacher in Columbia, Missouri, and a participant in Project EXTRA (Enhancing the Teacher's Role in Assessment—the *x* is the unknown), funded in part by the National Science Foundation. The authors have subsequently used a version of this as a test item on a midterm exam for preservice secondary teachers. The item reads,

> Compare and contrast the meaning of the "–" in the following expressions:
>
> a) -3
> b) 4^{-3}
> c) $\tan^{-1}(3)$
> d) $f^{-1}(3)$

The following is a student's response that is representative of more than half of the responses we have received over the last several years.

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The meanings of the "–" in the expressions have nothing to do with each other. In –3 it means "negative" or "opposite" of three. Since 3 is on the right side of 0, –3 is on the left side of 0. In the expression 4^{-3} it means to write the 4^3 in the denominator instead of the numerator. It's kind of like using the reciprocal. In $\tan^{-1}(3)$ it could mean to put the tan in the denominator, but usually it means to take the arctan 3. $f^{-1}(3)$ just means to take the inverse function. Mathematicians use the negative sign to mean many different things, but it is usually obvious what the teachers want by the context or what you have been studying recently.

Clearly, this student has not made the connection to the larger concept of inverse. This is a common problem among high school students, even those who get very good grades in mathematics classes, as did the teacher who wrote the following on the board while teaching a trigonometry class:

Solve: $\tan x = 4/3$, where x is measured in degrees.

$$\frac{\tan x}{\tan} = \frac{1.33}{\tan}$$

Divide both sides by tan to get an answer of 53 degrees.

This teacher did not actually "divide" by tan, but that is what he said, and his pupils were told that "in this case" instead of pressing the divide button on the calculator what they really should do is type "2ND" or "shift" tan.

These examples illustrate the great divide between procedural and conceptual understanding of inverse. A great many of the problems we study involve the concept of inverse, several of which will be discussed in this article. It is important to assign terms, such as *opposite, reciprocal,* and *arctangent,* to differentiate among the specific inverses of various types; but as teachers, we must also call attention to the fact that each involves reversing the operation with which it is associated, which is the big idea behind inverses. While many students find success in mathematics classes by merely learning the vocabulary and what they have to "do" without ever really understanding the connections to the bigger picture, there are a great many other students who struggle as they work to memorize what they perceive to be a long list of unrelated terms. It is valuable for both groups of students to make the connections to the bigger idea for at least two reasons. First, the bigger idea of inverse should be presented as *one* powerful tool (as opposed to many) for solving problems. Second, because it is a recurring fundamental concept, it is also a vehicle for learning more mathematics at a deeper level.

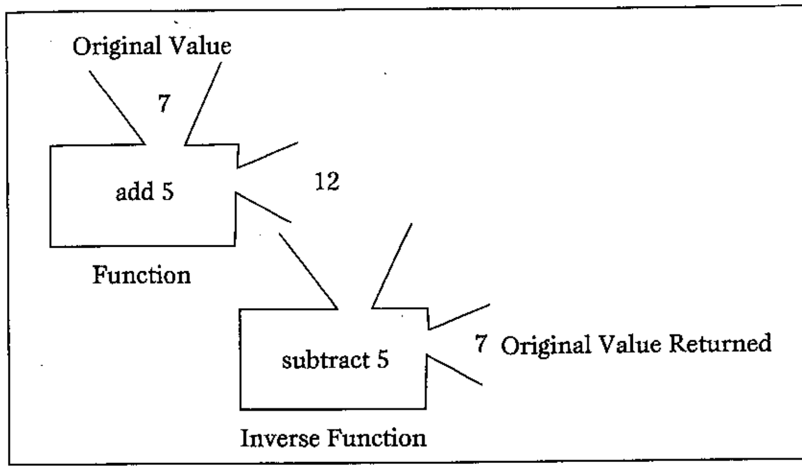

Fig. 1 Consecutive function machines can model additive inverses.

Different textbooks define *inverse* in different ways, depending on the context in which it is being used; but as an overarching concept, an inverse is anything that reverses or undoes a process. In elementary school, students learn that subtraction is the inverse operation of addition because if they add 5 to a number and then subtract 5 from that sum, they will get back to the original number. This is often modeled using consecutive function machines (see **fig. 1**). Subtraction undoes addition, and vice versa. Similarly, multiplication and division undo each other, as long as division by zero is not required.

Why is understanding the process of "undoing" so important? A fourth-grade child was playing a computer game that asks the player to figure out what number the $\triangle$ stands for in an equation like $4 \times \triangle - 7 = 17$. When asked how she figured it out, she said, "I just undid the problem."

"Show me," we said.

"Well, this means to multiply. So 4 times the triangle has to equal 24."

"Why 24?"

"Because you have to be able to subtract 7 from the 4 times the triangle, and if you just made 4 times the triangle equal 17 [pause as she looks back at the screen] or something like that, then when you subtracted 7 your answer would be too small."

"Okay, but how did you figure out it had to be 24?"

"I don't know. [pause] I just added 7 and 17 and that worked. I knew it had to be bigger than 17 so I would have room to subtract 7 and still get the right answer."

"Okay, so what answer did you get?"

"6."

"Are you sure that is the right answer?"

"Yeah, because if you take 4 times 6 you get 24, and when you minus 7 you get 17."

"Great! Thank you for explaining that to me."

This child, whether she realized it or not, was using the problem-solving strategy of working backward, which ultimately is using the concept of

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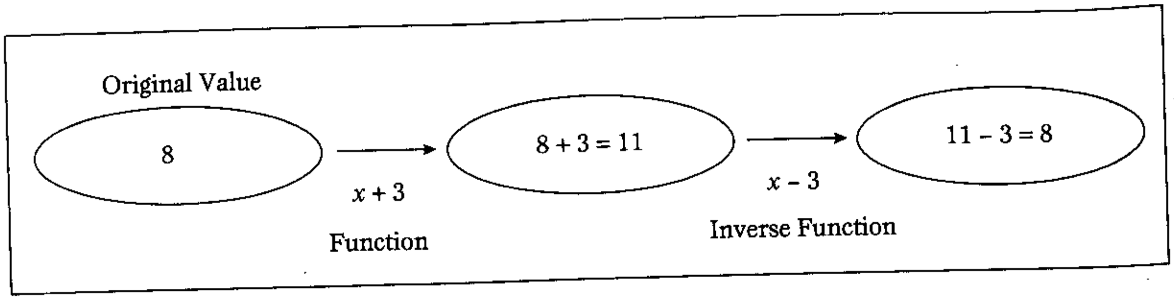

Fig. 2 Reversing addition by subtraction

inverse. I have watched several elementary-level children use similar thinking when they try to figure out how much something is worth. Unfortunately, some of these same children struggle as they try to solve equations like $4x - 7 = 17$ by trying to follow an algorithm they learned in class; they do not see the connection.

When students enter the late middle and secondary levels, in addition to talking about inverse operations, they are exposed to more formal definitions of inverse. For example, given an operation * on a set A, the * inverse of $a \in A$, if it exists, is the element $b \in A$ such that $a*b = c$, where c is the identity element of A. More specifically, $3 + -3 = 0$, where 0 is the identity element for addition. Therefore, –3 and 3 are additive inverses, and thus if you have any value x and add 3 to it, adding –3 to the resulting sum will "undo," or reverse, the addition, thus restoring the expression to its original value x (see **fig. 2**). For all numbers, $x + v + -v = x$ and $x + -v + v = x$. Later, we use this understanding to solve equations by applying inverse operations to both sides of an equation to isolate the variable in question. Solve $x + 3 = 7$ by adding –3 to both sides of the equation, which yields $x + 3 - 3 = 7 - 3$, so $x = 4$.

Students may struggle with solving equations at any grade level. Their difficulties are often rooted in their lack of understanding of the concept of inverse. To address this, write an equation similar to those above, ask students what is being done to the variable, and talk about solving by undoing or reversing the process (as was done in the example with the fourth grader). Then tie that to the term *inverse,* and talk about what they already know about inverses, which may vary greatly among students. The analogy of packing a suitcase is applicable. If you packed shirts first and then pants on top of those, the pants must come out first and then the shirts when the suitcase is unpacked. Once students understand this, they can rework the problem by "undoing"—but this time, using the notation of applying the appropriate inverses to both sides of the equation. Our students indicate that they understand when they say something like "Oh, it's the same thing I just did!" or they just sit back in their chairs and smile. Some students need more convincing, which is good, so we should let them choose problems to solve.

As we can see from these encounters with students, a critical element in understanding the idea of inverses is good discourse. If teachers do not ask questions at varying levels of the taxonomy, students do not develop understanding deeper than performing steps in an algorithm. What is the reverse operation? What is another way of writing this? How is this similar to . . . ? Can you give me another example? Asking questions such as these lead to discussions that help students understand rather than memorize.

Problem-solving strategies such as solving a simpler problem, looking for a pattern, and drawing a diagram are useful in helping students develop understanding of inverses. Using analogies such as packing a suitcase for a trip can make the inverse operation seem less abstract. The bottom line is this: Simply telling does not work. As with many opportunities for teaching and learning, understanding comes when students can ask and answer questions that will help them construct new knowledge from prior knowledge.

MAKING THE CONNECTIONS

We must help students understand the concept of inverse by discussing the connections in different circumstances. For example, knowing that 1 is the identity element for multiplication and $(4^{-3})(4^3) = 1$, we see that 4^{-3} is the multiplicative inverse, or reciprocal, of 4^3. Using the more general concept, 4^{-3} and 4^3 are multiplicative inverses; thus, if you have any value x and multiply it by 4^3, then multiplying the result by 4^{-3} will undo the multiplication and restore the expression to its original value x (see **fig. 3**). In general, $x(v^j)(v^{-j}) = x$ and $x(v^{-j})(v^j) = x$ for all values of x, j, and v, except for the restriction that $v \neq 0$. Note the restriction of the conditions under which this inverse may occur.

It is not uncommon to have restrictions under which an inverse occurs. For example, a function must be one-to-one in order for its inverse function to exist. When students study inverse functions, the definition for inverse is restated in terms of the composition of functions, but the overall concept remains the same: Under certain restrictions that allow an inverse to exist, the inverse reverses the process of the original operation. Given any input a, a function f produces an output of $f(a)$. Then f^{-1} is the inverse

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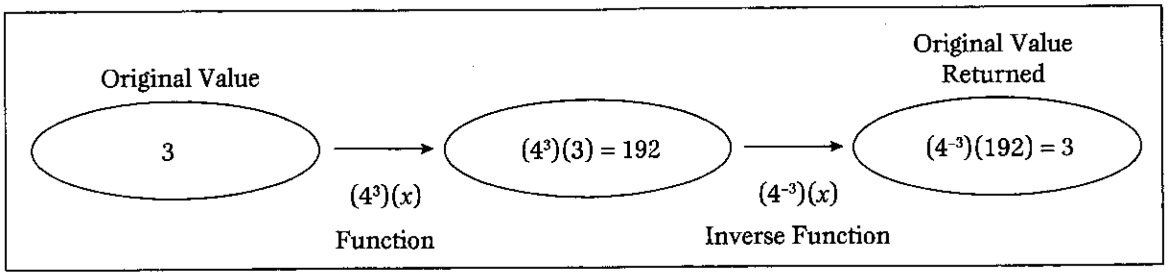

Fig. 3 Reversing multiplication using reciprocals

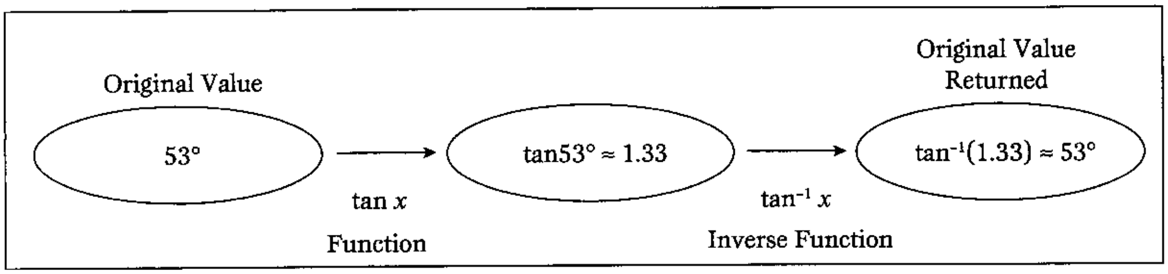

Fig. 4 The inverse function "undoes" what the function "does."

of f if and only if $f^{-1}(f(a)) = a$ and $f(f^{-1}(a)) = a$. Thus, f^{-1} reverses the process defined in the original function. In fact, one method for determining the inverse of a one-to-one function is first to write out, in words, the operations that are performed on the independent variable x, then rewrite these steps in reverse order using the inverse operations, and finally write the corresponding symbolic equation. For example, $f(x) = (2x - 5)^3$ means to multiply x by 2, subtract 5, then cube the resulting difference. Reversing the steps and using inverse operations we take the cube root, add 5, and divide by 2, so

$$f^{-1}(x) = \frac{\sqrt[3]{x}+5}{2}.$$

We can verify that this is an inverse function by finding

$$\begin{aligned} f^{-1}(f(x)) &= \frac{\sqrt[3]{(2x-5)^3}+5}{2} \\ &= \frac{2x-5+5}{2} \\ &= \frac{2x}{2} = x \end{aligned}$$

and

$$\begin{aligned} f(f^{-1}(x)) &= \left(2\left(\frac{\sqrt[3]{x}+5}{2}\right)-5\right)^3 \\ &= \left(\sqrt[3]{x}+5-5\right)^3 \\ &= \left(\sqrt[3]{x}\right)^3 = x. \end{aligned}$$

The inherent nature of functions and their domains and ranges may require restrictions to a domain of a function or its inverse in order to maintain the conditions that $f^{-1}(f(a)) = a = f(f^{-1}(a))$ for all a under consideration. For example, arctan x is the trigonometric inverse function of tan x because $\tan^{-1}x$, or arctan x, reverses the process posed by tan x. When we input an angular value (degrees or radians), tan x outputs a ratio value, and when we input that ratio value back into arctan x, this function outputs the angular value within the principal domain of tangent, the interval (90°, –90°), or

$$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Note that since tangent is a periodic function, inverse functions do exist outside the principal domain, but these are rarely discussed in the secondary school setting. Within the principal domain of tangent, $\tan^{-1}x$ reverses, or undoes, the function of tan x and vice versa, so that $\tan^{-1}(\tan x) = x$ and $\tan(\tan^{-1}x) = x$. If a student understands this relationship, then instead of writing

$$\frac{\tan x}{\tan} = \frac{1.33}{\tan}$$

as illustrated earlier, the student would solve tan x = 1.33 by applying arctangent to both sides of the equation. Thus, $\tan^{-1}(\tan x) = \tan^{-1}(1.33)$, and so x = 53° (see **fig. 4**).

To reinforce the inverse relationship, arctan x is often written as $\tan^{-1}x$. But there are other functions for which the notation does not explicitly indicate this relationship. The fact that some operations are inverses of each other is surprising if students do not fully understand the operations.

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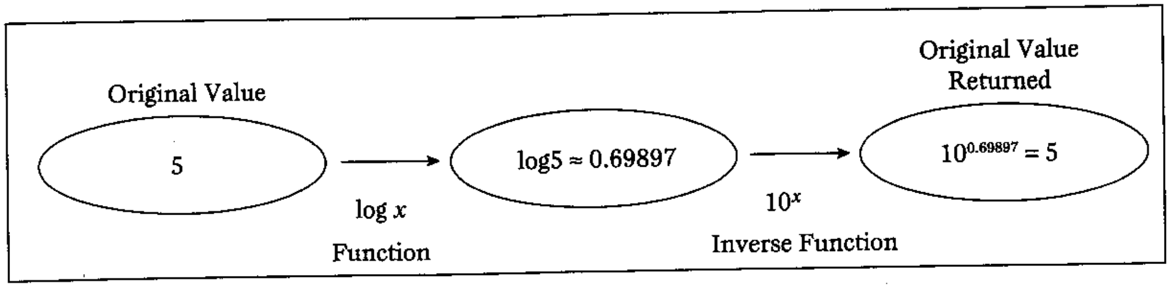

Fig. 5 Exponentiation "undoes" what the log function "does."

Exponentials and logarithms are commonly misunderstood in this way. For example, if $x > 0$, then $10^{\log x} = x = \log(10^x)$, so $\log x$ and 10^x are inverses of each other for $x > 0$. Conceptually, in the expression $10^{\log x}$, 10 is raised to a power. That power of 10 is a logarithmic expression that represents another power of 10 or log base 10 of x. Symbolically, if x were the input to the logarithm function such that $\log x = y$, then $10^y = x$. Applying the exponentiation of 10 on both sides of the equation $\log x = y$ yields $10^{\log x} = 10^y$, which equals x, showing that $10^{\log x} = x$. The logarithm property $\log(x^a) = a \log x$ simplifies $\log(10^x)$ to $x\log(10) = x(1) = x$, thus showing that $\log x$ and 10^x are inverses of each other for $x > 0$ (see **fig. 5**). Once students understand that these functions are inverses, they should have no trouble applying them to solve equations such as $10^x = 3$. Applying common log, the inverse function of the exponentiation base 10, to both sides of the equation results in $\log(10^x) = \log(3)$, so $x = \log 3 \approx 0.477$. Note: There is no y such that $10^y = x$ when $x \leq 0$. Similar relationships can be shown for e^x and $\ln x$.

THE MATHEMATICAL POWERS OF INVERSE

Understanding inverse as a way of breaking something down by reversing the "packing" order of the relationship enables a greater number of students to wield mathematical power. In *Principles and Standards for School Mathematics* (NCTM 2000), the Connections Standard calls for programs that help students "recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; [and] recognize and apply mathematics in contexts outside of mathematics" (NCTM 2000, p. 64). The Communication Standard states that students should be able to "use the language of mathematics to express mathematical ideas precisely" (NCTM 2000, p. 60). Students do need to understand the differences between reciprocals and when it is appropriate to use each. However, discussing reciprocals as multiplicative inverses and opposites as additive inverses leads to much greater understanding of them and the power of inverses. Questions that direct students to "compare and contrast the meanings of the words *reciprocal* and *opposite*" force respondents to look for connections and help teachers assess the comprehension of the larger concept. Understanding that these terms and the concepts they name are part of a larger fundamental concept helps students recognize the connections among these terms and other terms involving inverse and helps them understand that they are interconnected, allowing them to create a "coherent whole."

CONCLUSION

For many students, mathematics is anything but coherent, and textbooks often do not make connections among concepts. As mathematics teachers, we must teach more than what is in the text and help students see and understand the big picture. Doing so will also help them recognize the concept and apply it in contexts outside of mathematics.

The exploration of the concept of inverse is of value because it helps students defend, refine, or reinforce their personal ideas about what inverse is or does. Exploration of the inverse also models for students that in mathematics we look for patterns, ask questions, and reason our way toward solutions and learn as much as possible along the way.

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Musser, Gary L., William F. Burger, and Blake E. Peterson. *Mathematics for Elementary Teachers: A Contemporary Approach.* 5th ed. New York: John Wiley & Sons, 2001.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000. ∞

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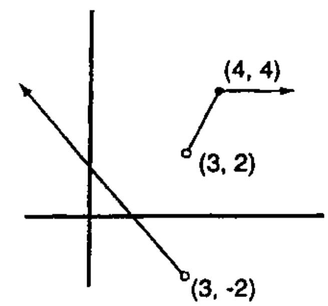
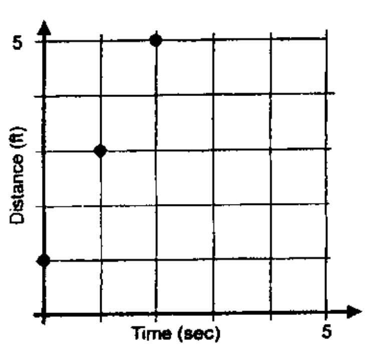
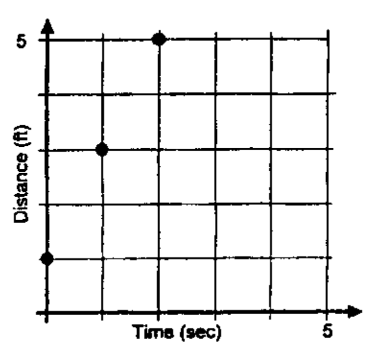
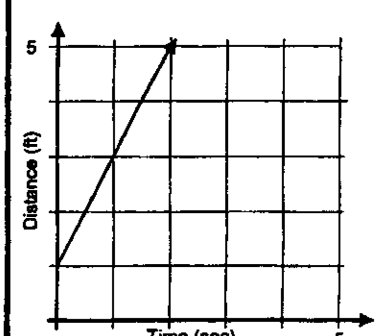
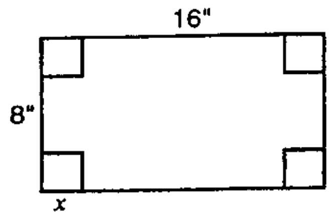
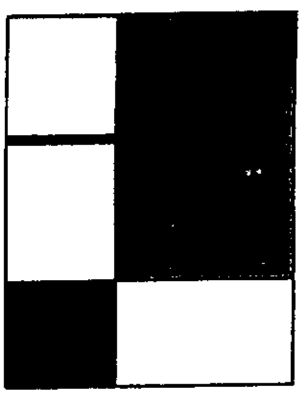
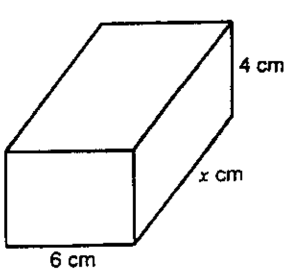
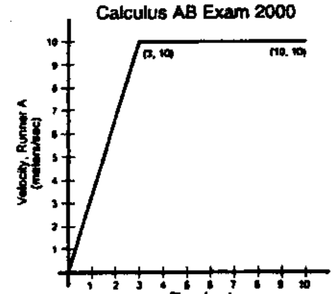
Concepts and Skills	6/7	7/8	Algebra 1
Domain and Range	Given the inputs (domain) {1, 3, 4.5, 10.75}, fill in the box to find the outputs (range). $\square + 2.5 =$ Outputs { ___, ___, ___, ___ }	Given the domain {-1, 0, 1, 2} for *a*, find the range for *3a-4*.	Give the domain and range for the graphed function. 
Factoring	Use the prime factorization of 18 and 24 to find their greatest common factor and least common multiple.	Find two integers having a product of -15 and a sum of 2	Factor $x^2 + 2x - 15$. Factor $2x^2 + 7x - 15$. Solve by factoring: $2x^2 + 7x + 3 = 18$
Functions	The graph below shows the number of feet Danny is from his friend as Danny walks down the hall. Explain the pattern for Danny's walk. If Danny continues to walk away from his friend in this same pattern, find Danny's distance from his friend after 5 seconds. 	The graph below shows the number of feet Danny is from his friend as Danny walks down the hall. Write a formula to find Danny's distance from his friend at any time *t*. If Danny continues to walk away from his friend in the same pattern, use the formula to find Danny's distance from his friend after 21 seconds. 	The graph below shows the number of feet Danny is from his friend as Danny walks down the hall. Write a function *d(t)* to find Danny's distance from his friend at any given time. Find *d(6.2)*. Explain the meaning of *d(6.2)*. 

<table>
<tr><th>Geometry</th><th>Algebra 2</th><th>Pre-Calculus</th></tr>
<tr>
<td>A piece of cardboard is to be made into a box by cutting out a square x inches wide on each corner and folding up the edges. Find the feasible domain for x.

</td>
<td>Give the domain and range for
$$f(x) = \sqrt{4x^2 - 9} + 2$$</td>
<td>Find the domain and range for
$$f(x) = \frac{\ln(2x^3 + e)}{2}$$</td>
</tr>
<tr>
<td>The area of a modern art painting is 15 square inches. Find the dimensions of the painting if the height is 7 inches more than twice the width.
</td>
<td>Solve by factoring.
$$2e^{2x} + 7e^{x} + 3 = 18$$</td>
<td>Solve by factoring:
$$2e^{2\sin x} + 7e^{\sin x} + 3 = 18$$</td>
</tr>
<tr>
<td>Write a function $f(x)$ to find the surface area of the box below in terms of x.

Find the value for x if $f(x) = 362$. Explain the meaning of your answer. Be sure to include units.

</td>
<td>A clown throws a ball straight up into the air. The height of the ball at a given time after leaving the clown's hand is shown in the table below.
<table>
<tr><th>Time (seconds)</th><th>Height (meters)</th></tr>
<tr><td>0.2</td><td>2.804</td></tr>
<tr><td>0.5</td><td>4.775</td></tr>
<tr><td>1.05</td><td>4.975</td></tr>
</table>
Write a quadratic equation $h(t)$ for the height of the ball at any given time.

Find $h(0)$ and explain the meaning of this answer in the context of this problem.</td>
<td>Calculus AB Exam 2000


Two runners (A and B), run on a straight track for $0 \leq t \leq 10$ seconds. The graph above shows the velocity of Runner A. The velocity, in meters/second, of Runner B is given by the function v defined by
$$v(t) = \frac{24t}{2t + 3}$$
Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.</td>
</tr>
</table>

<table>
<tr><th>Concepts and Skills</th><th>6/7</th><th>7/8</th><th>Algebra 1</th></tr>
<tr><td>Limits</td><td>A plant is growing in the following pattern. If the pattern of growth continues, what is the maximum height of the plant?

<table><tr><th>Month</th><th>Height (ft.)</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td></tr><tr><td>3</td><td>3.5</td></tr><tr><td>4</td><td>3.75</td></tr><tr><td>5</td><td>3.875</td></tr><tr><td>6</td><td>3.9375</td></tr></table></td><td>$\frac{1}{2}+\frac{1}{4}=$
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=$
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=$
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=$

Based on the pattern, can you predict the sum of the first 10 terms?</td><td>$f(x) = \frac{3x-1}{x+1}$
$f(10) =$
$f(100) =$
$f(1000) =$
As x gets large, $f(x)$ approaches what number?</td></tr>
<tr><td>Optimization</td><td>Given three pieces of string each 12 inches long, form a triangle, square, and circle. Which figure has the maximum area?</td><td>Given a rectangle with a perimeter of 24 inches, determine the dimensions that maximize the area.</td><td>The height of a thrown ball is given by $h(t) = -16t^2 + 48t$.

At what time does the ball reach its maximum height?</td></tr>
<tr><td>Transformations</td><td>Plot the points (1,1), (4,1), (5,3) on a coordinate plane and connect the points to form a triangle. Using patty paper, translate the triangle 8 units to the left. Write the coordinates of the vertices for the translated triangle.</td><td>Plot the points (1,1), (4,1), (5,3) on a coordinate plane and connect the points to form a triangle. Write a formula that will translate the point (x,y) 8 units to the left.</td><td>Translate the line, $y = 2x$ eight units to the left. Write the function for the translated line in slope-intercept form.

How does the equation $y=2(x+8)$ relate to the function you wrote and the translation you performed?</td></tr>
</table>

<table>
<tr><th>Geometry</th><th>Algebra 2</th><th>Pre-Calculus</th></tr>
<tr>
<td>A regular n-gon is inscribed inside a circle with a radius of 10. As n increases, the perimeter must be less than _____ ?</td>
<td>

$$f(x) = \frac{3x-1}{x+1}$$

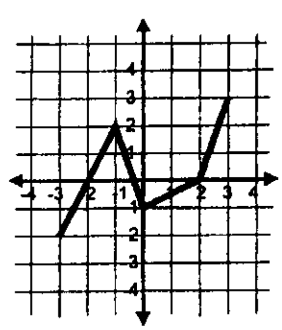
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>- .5</td><td></td></tr>
<tr><td>- .7</td><td></td></tr>
<tr><td>- .9</td><td></td></tr>
</table>

Describe what is happening to the value of the function as x gets closer to -1.</td>
<td>

$$\lim_{x \to -1} \frac{3x-1}{x+1}$$

$$\lim_{x \to \infty} \frac{3x-1}{x+1}$$

</td>
</tr>
<tr>
<td>Squares are cut from the corners of an 8.5 inch by 11 inch paper and sides are folded to make an open-topped box. How much should be cut from each corner to produce a box of maximum volume?</td>
<td>Given a box with a square base and a volume of 48 cubic inches, find the dimensions that minimize the surface area.</td>
<td>A rectangle is inscribed in an ellipse centered at the origin with axes of lengths 8 and 6. Find the maximum area of the rectangle.</td>
</tr>
<tr>
<td>Translate the circle $x^2 + y^2 = 25$ eight units to the left. Write the equation for the translated circle.</td>
<td>Given the graph of the function $f(x)$:



Graph the transformation $f(x+8)$. Write a piecewise function for $f(x)$ and $f(x+8)$.</td>
<td>Write in function notation the equation for the graph of $f(x) = e^x$ translated eight units to the left.

Simplify the equation so that it is expressed in terms of $f(x)$.</td>
</tr>
</table>

Concepts and Skills	6/7	7/8	Algebra 1
Rate of Change	Which of the following are examples of rates? a) miles you have driven b) the number of elephants you capture in 3 video games c) how may people walk in a park per day d) glasses of milk you drink e) gallons of ice cream	The water level in a tank is dropping at the rate of 3 inches every 4 minutes. At what rate is the water level dropping in feet per hour?	Which function or functions have a constant rate of change? (see tables A–D below)
Sequences and Series	What is the sum of the missing terms? 78, 38, 18, 8, ___ , ___	What is the 7th term of the sequence? $x+6, x-5, x+4, \ldots$	What is the sum of the infinite series? $8 + 4 + 2 + 1 + \ldots$
Probability	You wish to toss two coins. List the possible outcomes in the sample space. Find the probability that two tails will occur.	Toss a die and a coin. A) Find the probability that a head and four will occur. B) Find the probability that a tail or a three will occur.	In a random sample of 100 people, 63 own computers, 38 own DVD players, and 15 own both. Represent the data in a Venn diagram. A) Find the probability that a randomly selected person owns a DVD but not a computer. B) Find the probability that a randomly selected person owns neither a DVD player nor a computer.

A

x	f(x)
1	1
2	3
4	5
8	7

B

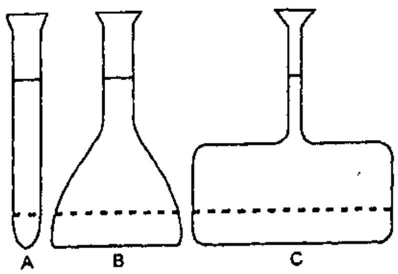
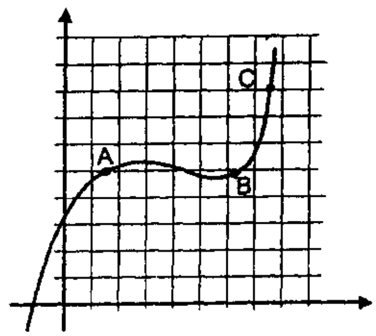
x	g(x)
1	3
2	3
4	3
8	3

C

x	h(x)
3	2
6	6
12	14
21	26

D

x	j(x)
3	0
6	1
9	4
12	9

Geometry	Algebra 2	Pre-Calculus
Water is pouring in at the same rate in all 3 containers. In which of the containers is the water level at the dark line rising the fastest? In which of the containers is the water level at the dotted line rising fastest? 	What is the letter of the point on the graph where the rate of change is the fastest? 	For $s(t) = 3t^2 + t - 2$, find the average rate of change for $s(t)$ for $[-1,3]$.
If the pattern below is continued, how many dots will be in the sixth figure? 	1, 3, 6, 10, 15, ..., a_n Write a formula for a_n.	$\frac{\sin x}{x} + \frac{\sin 2x}{x} + \frac{\sin 3x}{x} + \frac{\sin 4x}{x}$ If x is a very, very small number, what is the approximate sum of the series?
A circle is inscribed inside a square with a side length of 5 units. What is the probability that a randomly selected point will be in the square but not in the circle?	How many tickets are possible in the Texas Lotto if a player may select six different numbers from the 55 available? Based on your answer, what is the probability that a particular six number combination will be selected for the Lotto winner?	A team is to play five games over the next week. The probability that they win any one game is .75. Assuming that the games are independent, use the binomial theorem to determine the probability that the team will win exactly four of the five games.

I have come to the frightening conclusion that I am the decisive element in the classroom. It's my personal approach that creates the climate. It's my daily mood that makes the weather. As a teacher, I possess a tremendous power to make a child's life miserable or joyous. I can be a tool of torture or an instrument of inspiration. I can humiliate or humor, hurt or heal. In all situations, it is my response that decides whether a crisis will be escalated or de-escalated and a child humanized or dehumanized.

Haim Ginott

Karen Heinz and Blair Sterba-Boatwright

The When and Why of Using Proportions

In our mathematics classes, we engage students in explorations involving similarity, relative growth and size, dilations, scaling, pi, constant rate of change, slope, speed, rates, percent, trigonometric ratios, probability, relative frequency, density, and direct and inverse variations. What do all of these (and many other) mathematics concepts have in common? They all involve ratios.

The importance of having a deep understanding of ratio and proportion has been well noted. Lesh, Post, and Behr (1988) declared proportional reasoning a pivotal mathematics concept, both the capstone of elementary mathematics and the foundation of upper-level mathematics. Confrey and Harel (1994) stressed that understanding multiplicative concepts (e.g., ratio) is critical for successful completion of middle school mathematics programs, which then separates students who are able to continue on to higher-level mathematics from those who cannot. Ratio and proportion are fundamental concepts, and yet a significant number of students are not able to solve problems that involve proportional reasoning. In this article, we highlight some of the fundamental components of the understanding of ratio and proportion and explore ways in which we can help our students develop this understanding.

This department consists of articles that bring research insights and findings to an audience of teachers and other mathematics educators. Articles must make explicit connections between research and teaching practice. Our conception of research is a broad one; it includes research on student learning, on teacher thinking, on language in the mathematics classroom, on policy and practice in mathematics education, on technology in the classroom, on international comparative work, and more. The articles in this department focus on important ideas and include vivid writing that makes research findings come to life for teachers. Our goal is to publish articles that are appropriate for reflection discussions at department meetings or any other gathering of high school mathematics teachers. For further information, contact the editors.

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THE SUE AND JULIE PROBLEM

We begin by posing a problem that Cramer, Post, and Currier (1993) gave to thirty-three preservice elementary teachers in a mathematics methods class:

> Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run? (p. 159)

We encourage you to solve this problem before reading further.

If you solved this problem by setting up a proportion, perhaps you did so because you know this article is about ratios and proportions. If so, we ask you to go back and read the problem again, looking for information that leads you to believe that setting up a proportion is an appropriate way to solve this problem.

To solve this problem correctly, one must note that Sue and Julie "were running equally fast" and understand what this indicates about the relationship between the number of laps each ran. If they are running equally fast, then Sue runs 1 lap (not

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3) for every lap that Julie runs. Therefore, because Julie ran only 3 laps when Sue had run 9 laps, Sue was and will always be 6 laps ahead of Julie. The correct answer, then, is that Sue has completed 21 laps (not 45) when Julie has completed 15 laps. Note that what remains constant in the problem is the *difference* between the number of laps each ran.

Preservice and practicing elementary and middle school teachers frequently solve this problem by setting up a proportion that gives them an answer of 45 laps. Cramer, Post, and Currier (1993) observed that thirty-two out of thirty-three preservice teachers used a proportion. We have posed this problem to hundreds of teachers with whom we have worked, and the vast majority of these teachers also have set up a proportion to solve this problem.

Understanding a proportion as an arrangement of four related quantities

What is it about the Sue and Julie problem that lures so many people into incorrectly using a proportion to solve it? It seems to fit the structure of a proportion problem with which students are familiar: It gives three quantities, asks for one unknown quantity, and pairs these four quantities in two ways: by person and by points in time. These are key aspects of the problem for our students, as indicated by their explanations. When our students have presented the proportions they used, their explanations have been focused on how to arrange the four quantities in the proportion according to the pairings of the quantities. Students stated that the numerators of the ratios can represent the number of Julie's laps (or the number of Sue's laps) or they can represent the number of laps at the first point in time (or at the second point in time). Students also emphasized that once a choice has been made, one must be consistent. For example, if one focuses on Julie and decides to put the number of her laps in the numerator of the first ratio, then one must put the number of her laps in the numerator of the second ratio as well.

Student responses to this task invite us to make some distinctions in what they understand about proportions. Students have learned how to arrange the four quantities in a proportion (for a directly proportional situation), and they explicitly attend to the placement of the four quantities when setting up a proportion. This knowledge has served them well in correctly solving a variety of problems that involve proportions. However, the students did not seem to be attending to the fact that a proportion is more than just a structural way to organize four quantities. They did not seem to be working from an explicit awareness that a proportion is an indication and a claim about the relative size of the quantities involved—there are multiplicative relationships involved that remain constant in significant ways. We will explore these constant relationships as we suggest ways to strengthen students' proportional reasoning.

Requiring students to justify their use of proportions

What can we as teachers do to help students develop an understanding of proportion that extends beyond seeing it as a way to organize four related quantities? Our procedure was consistent with the overarching approach to teaching of always asking students to justify their choice of operation by referring to the given situation (Thompson et al. 1994): We asked students to explain why they chose to use a proportion. An analysis of their responses to the Sue and Julie problem indicated that we should require students to provide answers that reflect deeper understanding than does this one, for example: "I was looking for the number of laps Sue completed when Julie completed 15 laps, so I put the number of laps Julie completed in the numerator of both ratios." We noted that such a procedural response is not satisfactory because it does not extend beyond making a claim that the four quantities are related in some way, which is an incomplete and inadequate understanding of proportion. We noted that the four quantities are indeed related, but in this case the constant relationship is a difference, not a ratio. Therefore, using a proportion is not appropriate.

We needed to require justifications in which students explain why it is appropriate to set two ratios equal to each other. In other words, students need to answer this question: What do we know about the quantities in the problem and the relationships between them that indicates that the two ratios in the proportion are indeed equal? Note that if we set two nonidentical ratios equal to each other, as in

$$\frac{A}{B}=\frac{C}{D},$$

this statement of equality involves two multiplicative relationships:

1. C is larger or smaller than A by the same factor that D is larger or smaller than B.
2. B is larger or smaller than A by the same factor that D is larger or smaller than C.

Depending on the problem's context, one of these relationships may be more meaningful than the other.

A brief example of how requiring a justification from students might play out in the classroom follows: If students wrote the proportion

$$\frac{3}{15}=\frac{9}{x}$$

to solve the Sue and Julie problem, the teacher could ask, "By setting up this proportion, what are we claiming about how fast Sue is running compared with how fast Julie is running?" Students who recognize that a valid way to solve this proportion is to multiply 15 by 3 (since 3 times 3 is 9) are generally able to explain that such a proportion indicates that Sue is running 3 times as fast as Julie. If students do not spontaneously refer to the problem's context to find out that this is not the case (since Sue and Julie are running equally fast), the teacher can ask a question such as, "What do we know about the situation that indicates that Sue is running 3 times as fast as Julie?" Again, the goal is to have students justify their use of a proportion in this context. When looking back to the problem to find information about the speeds, students notice that Sue and Julie are running equally fast. The discussion can then focus on how this information is useful in determining the answer to the problem. While experiencing varying degrees of difficulty in solving this problem, our students have been able to realize that the difference between the number of laps each girl runs remains constant. Later in this article, we will return to a discussion of how to engage students in an exploration of the constant relationships in this problem and the key role these differences play.

THE ROOM PAINTING PROBLEM

We gained further insight into key aspects of student understanding of ratio and how it might be supported by examining student responses to another problem we posed to elementary and middle school preservice and practicing teachers:

> Bill can paint one of the new hotel rooms in 6 hours, and Mary can paint the same room in 4 hours. If they worked together, how long would it take them to paint the room?

Again, we encourage you to solve this problem before reading further.

This type of problem is often found in algebra books, along with a proposed solution method similar to the following: If Bill can paint the room in 6 hours, then his room-per-hour rate is 1/6. Similarly, Mary's room-per-hour rate is 1/4. Therefore, the number of rooms they can paint together in t hours can be expressed as $(1/6)t + (1/4)t$. To find how long it will take Bill and Mary, working together, to paint one room, set that expression equal to 1 and solve for t to conclude that they can paint the room in 12/5 hours, or 2 hours and 24 minutes.

Analyzing the understanding that underlies meaningful implementation of this solution method can be useful. However, we have rarely (if ever) observed preservice teachers using this method spontaneously, so we do not discuss it here.

Identifying a ratio

An approach to solving this problem that we have regularly observed among preservice teachers is the following: Because it takes Bill 6 hours to paint the room, he can paint 1/6 of the room in 1 hour. Similarly, Mary can paint 1/4 of the room in 1 hour. Therefore, if they work together, they can paint 1/6 + 1/4, or 5/12, of a room in 1 hour and then 10/12 of a room in 2 hours.

Most preservice teachers we have worked with who are successful up to this point are not able to complete the problem. They know that the answer is between 2 and 3 hours and many estimate that it is less than 2.5 hours, but they do not know how to find the exact answer.

We find this inability to complete the problem significant. These teachers are able to identify a ratio (5/12 of a room per hour), but having this ratio (whether or not they explicitly call it a ratio) does not seem to suggest to them either of two solution methods: using the ratio to set up a proportion or writing a linear equation of the form $y = mx$. The reasoning behind those two solution methods could proceed as follows:

Proportion method. We know that Bill and Mary, working together, can paint at a rate of 5/12 of a room per hour. Assuming that this rate will remain constant, we can write a ratio that expresses the relationship between the amount of room (1, for one full room) and the amount of time it takes to paint the full room (say, t) and set that ratio equal to

$$\frac{5}{12}:1$$

to get the proportion

$$\frac{5}{12}:1=1:t.$$

Then, we can solve that proportion for t and conclude that it will take Mary and Bill 12/5 hours to paint the room.

Linear equation method. Mary and Bill can paint 5/12 of a room per hour by working together. We know this rate remains constant, so the amount of room painted at any point in time will always be 5/12 as large as the amount of time it takes to paint that portion of room (i.e., the slope of a time versus amount of room line is 5/12). Thus, we can express the relationship between time, t, and amount of room, R, as $R = (5/12)t$. We can then solve for t when $R = 1$ to determine that it would take Bill and Mary 12/5 of an hour to paint one room.

Using the building-up strategy

The teachers with whom we have worked knew that if Bill and Mary could paint 5/12 of a room

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per hour, they could paint 10/12 of a room in 2 hours. It was a trivial matter for them to coordinate an increase of both quantities in the ratio by the amounts in the ratio (i.e., they increased the portion of the room by 5/12 and the length of time by 1 hour). Repeating this coordinated adding of the two quantities in the ratio as many times as needed has been referred to as the *building-up strategy* (Hart 1981; Kaput and West 1994; Lamon 1994).

Posing more challenging problems

If the question in the problem had been either "How much of the room can Bill and Mary paint in 3 hours?" or "How long will it take Bill and Mary to paint 5 rooms [60/12 rooms]?" it is likely that many more of our students would have successfully completed the problem because these two questions can be answered using the building-up strategy. Students would have been able to solve those problems on the basis of their current understanding. Therefore, in order to challenge students to develop greater understanding, we find it useful to present them with problems (e.g., the Room Painting problem) in which the numbers are such that the building-up strategy does not lead to an exact solution.

We next suggest ways in which teachers can have students explore problems such as the Room Painting problem to promote greater understanding of the interrelated concepts of proportional and linear relationships.

Promoting understanding of relationships that remain constant

When thinking about how to challenge and extend students' understanding, we want to focus and build on what they already know (NCTM 2000; Simon et al. 2004) and provide opportunities for *them* to build on what they know. When solving the Room Painting problem, students identified an important relationship—5/12 of a room could be painted in 1 hour—and they were able to use the building-up strategy to determine that 10/12 of a room could be painted in 2 hours. Therefore, we could ask students to make a two-column table relating the number of hours, t, it takes to paint the room and the portion of a room painted, R. An example is shown in **table 1**. We can also ask students to graph the relationship between the two variables. A sample graph is shown in **figure 1**.

With these two representations of the variables involved, we can ask a variety of questions with the goal of having students identify key relationships and explain why those relationships remain constant in the problem's context. Some useful questions include the following:

Table 1 Values for the Room Painting Problem

t	R
1	5/12
2	10/12
3	15/12
4	20/12
5	25/12
6	30/12

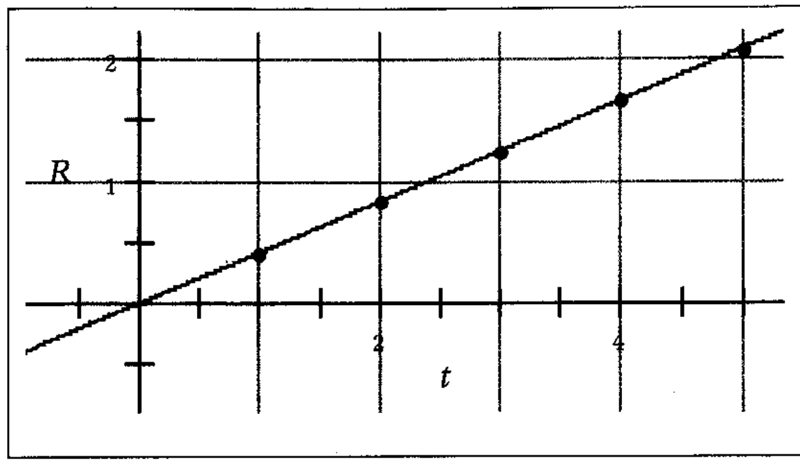

Fig. 1 Graph for the Room Painting problem

1. What relationships are there between and among the values shown in **table 1**?
2. How can we represent those relationships symbolically?
3. If we knew only one of the two values in any row in the table, could we find the other value? If so, how? (Remember to include values other than whole numbers for the amount of time and values other than integer multiples of 5/12 for the amount of room.)
4. Does it make sense to connect the points on the graph in **figure 1**? Why or why not?
5. If we connect the points on the graph, what do the intermediate points indicate? How can we find the coordinates of those points? When would it be useful to do so?
6. What is the equation of the line in the graph? How does the equation relate to the problem's context?

This type of activity engages students in exploring the key relationships in the problem by having them generate additional values for the variables in the problem and then make and analyze multiple representations of those quantities. Further, as students identify key relationships, they may symbolize these relationships by using ratios, proportions, and equations of the form $y = mx$.

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Table 2 Values for the Sue and Julie Problem

J	*S*
0	6
3	9
6	12
9	15
12	18
15	21

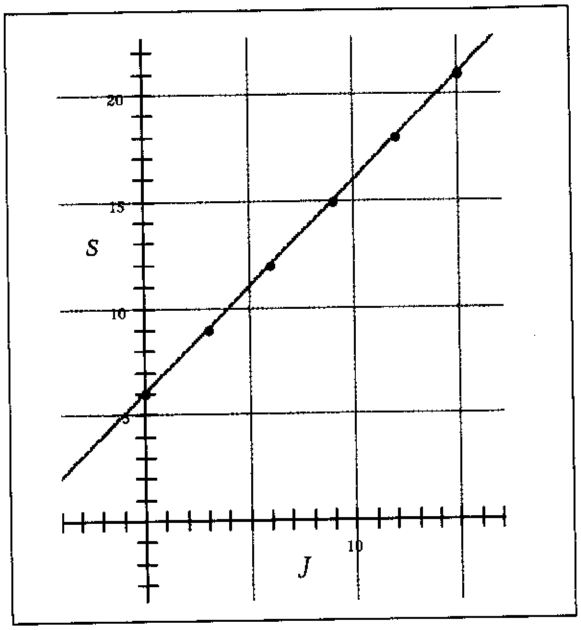

Fig. 2 Graph for the Sue and Julie problem

This same approach can be useful in exploring the Sue and Julie problem. The table and the graph that students might generate are shown in **table 2** and **figure 2,** respectively. Using the table and the graph, students can write an equation to express the relationship between the number of laps Sue ran, S, and the number of laps Julie ran:

J: $S = J + 6$.

After students have generated and analyzed these other representations of the Sue and Julie problem, teachers can ask students to look for similarities and differences between that problem and the Room Painting problem. For example, the y-intercept for the Room Painting problem is 0, as are the y-intercepts for all lines representing proportional situations, whereas the y-intercept for the Sue and Julie problem is 6, the number of laps that Sue had completed before Julie started running. These types of explorations can be particularly useful because they allow students to make distinctions between situations involving differences and those involving ratios, a key component of an in-depth understanding of ratio and proportion.

SUMMARY

Our students may be successful in solving a variety of problems involving ratio and proportion, yet their understanding of ratio and proportion may be incomplete. Students may be able to identify a ratio in a problem, but they may not know that the ratio can be the basis for writing a proportion or that it gives them a slope that can be used to write an equation of the form $y = mx$. This understanding must develop over time. Some suggestions for promoting and supporting the development of students' understanding of ratio and proportion follow:

1. Engage students in solving problems involving ratios that *cannot* easily be solved by using the building-up strategy.
2. Provide a variety of problems for students to solve that will require them to make decisions about when to use proportions.
3. When students do use a proportion, ask them to justify using it. Require a justification that includes statements about the multiplicative relationships involved, because these are the defining characteristics of a proportion.
4. Ask students to identify relationships that remain constant in various situations and to explain their meaning within the problem's context.
5. Have students symbolize the relationships and equalities they find by writing expressions involving differences or ratios and by writing proportions and linear equations of the form $y = mx$ or $y = mx + b$, as appropriate.

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Syllabus Scavenger Hunt

1. Which of the four tests you will take will <u>not</u> count towards your course grade?

   ___ ___ ___ -- ___ ___ ___ ___

2. You will keep a daily record of your thoughts and experiences during the program in your

   ___ ___ ___ ___ ___ ___ ___.

3. Each Wednesday, a speaker will present a talk and lunch will be provided during

   ___ ___ ___ ___ ___ ___ ___ ___ ___.

4. Participants will work in pairs to develop and present a unit called a(n)

   ___ ___ ___ ___ ___ ___ ___ ___

   ___ ___ ___ ___ .

5. During the team share, you and a partner will share a shortened version of a successful

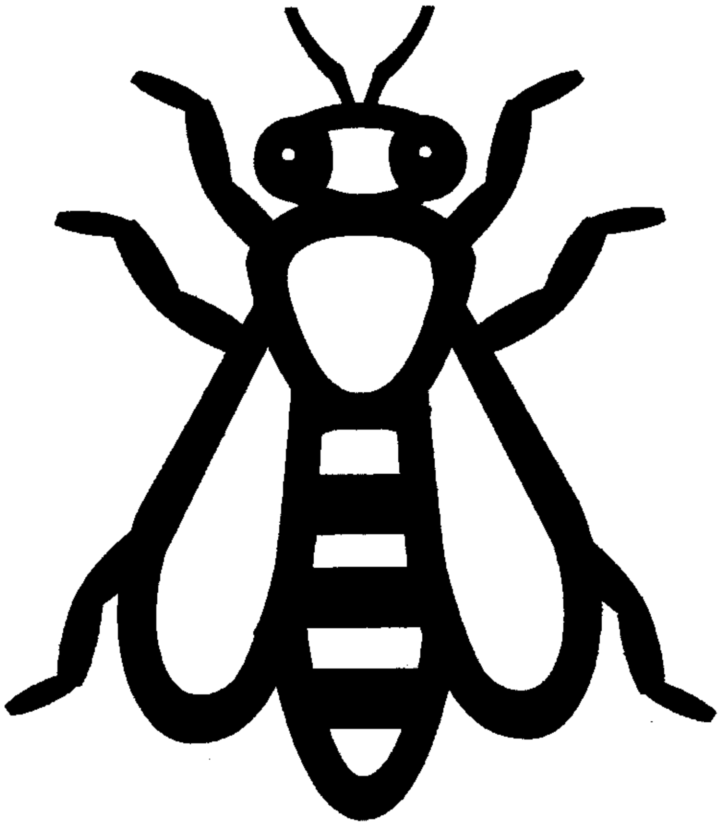
   ___ ___ ___ ___ ___ ___ ___ ___ .

Unscramble the circled letters to find out your first material to receive today.

Bring your answer to Julie who will tell you where you can collect your item.

Answer:

___ ___ ___ ___ ___ ___ ___ ___ ___ ___



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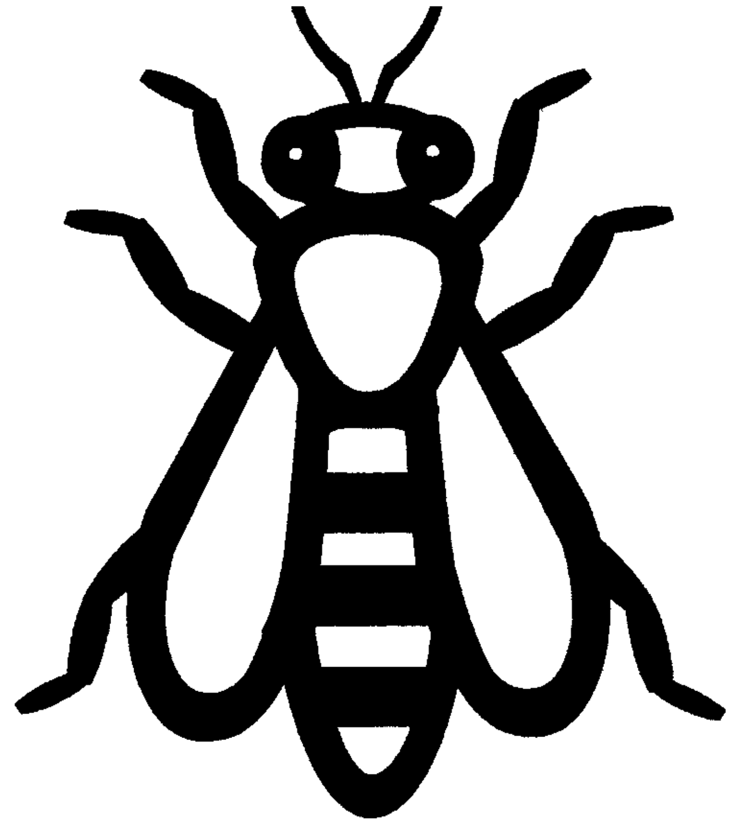

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Zero Out

A game for 2 – 4 players.

Face up version:

Dealer shuffles standard 52-card deck. Starting with the player to the left, passes out cards singly until each player has five cards face up on the table. Turn the next card face-up in the middle. Each ace is worth 1 (not high in this game), each 2 through 10 is worth that number, jacks count for 11, queens are 12, and kings are 13. The black cards have positive values; red cards have negative values.

The first player to the dealer's left looks at his/her cards and tries to use as many as possible to get the numbers to *add up* to the value of the card in the center of the table. While you are learning the game of Zero Out, other players may "help" by suggesting cards or card combinations. Note: including a black and red card with the same number showing will get rid of more cards but does not change the value of the discarded cards. When discarding, the card(s) must be added in full view of all the players so that the matching value may be verified. If a player cannot discard, his/her turn is over and that person receives one card from the dealer. Play passes to the left.

This is a game of luck because it depends on the cards you are dealt. It is also a matter of skill in that the player who is able to use a card, or cards, to match the center will discard those in such a way that the card (s)he now puts on top of the discard pile is chosen so as to be difficult for the next person to make.

The player who is out of cards first (value of zero!) is the winner and becomes the dealer for the next round.

Regular version:

Dealer shuffles standard 52-card deck. Starting with the player to the left, passes out cards singly until each player has five cards to pick up for their hand. Turn the next card face-up in the middle of the table. Each ace is worth 1 (only low in this game), each 2 through 10 is worth that number, jacks count as 11, queens are 12, and kings are 13. The black cards are assigned the positive point values while the red cards have negative point values.

The first player to the dealer's left looks at his/her cards and tries to use as many as possible to get the numbers to *add up* to the value of the card face up in the center. When discarding, the card(s) must be added in full view of all the players so that the matching value may be verified. If a player cannot discard one, or more cards, then his/her turn is over and (s)he receives one card from the dealer. Play passes to the left.

This is a game of luck because it depends on the cards you are dealt. It is also a matter of skill in that the player discarding chooses which card to put on top of the discard pile from those cards being discarded.

The player who is out of cards (value of zero!) first is the winner and becomes the dealer for the next round.

Advanced version: Make up your own variation.

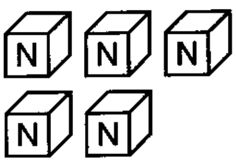
N. Leveille

The Cube Cruncher 1

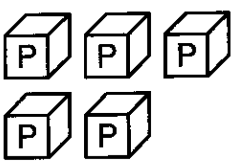
The *Start With* cubes are given. Apply the *Cube Cruncher Rule* to find the *End With* cubes. Draw the *End With* cubes. Positive Cubes [P] are pink or purple. Negative Cubes [N] are any other color.

Hint: One Positive Cube [P] and one Negative Cube [N] equal zero. They can be removed.

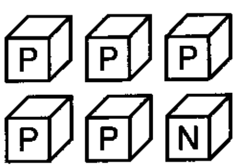
START WITH | **CUBE CRUNCHER RULE** | **END WITH**

1. 

Subtract 4 [N]

2. 

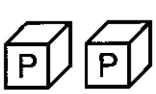
Add 2 [P]

3. 

Add 2 [N]

4. 

Add 5 [P]

5. 

Add 4 [N]

The Cube Cruncher 6

The *Start With* cubes are given and the *End With* cubes are given. Look at the *Start With* cubes and the *End With* cubes, then write the *Cube Cruncher Rule*. Positive Cubes [P] are pink or purple. Negative Cubes [N] are any other color.

START WITH	CUBE CRUNCHER RULE	END WITH
1. P P	?	P P P P
2. N N	?	N N N
3. P P P	?	P
4. P P	?	N N N
5. N N N	?	P P P

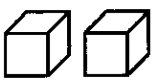
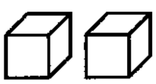
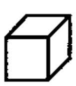
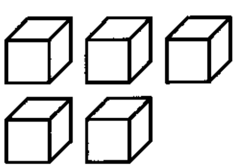
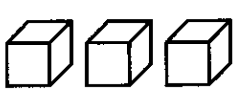
The Cube Cruncher 9

The *Start With* cubes are given and the *End With* cubes are given. Look at the *Start With* cubes and the *End With* cubes, then write two possible *Cube Cruncher Rules.* Positive Cubes $\boxed{P}$ are pink or purple. Negative cubes $\boxed{N}$ are any other color.

	START WITH	CUBE CRUNCHER RULE	END WITH
1.	P P P P P P	? or ?	P P
2.	N N N N N	? or ?	N
3.	P P P	? or ?	P P P P

The Cube Cruncher 13

The labels on the *Start With* cubes are missing. The *Cube Cruncher Rule* is given and the *End With* cubes are given. Look at the *End With* cubes and the *Cube Cruncher Rule*. Label the *Start With* cubes. Positive Cubes [P] are pink or purple. Negative Cubes [N] are any other color.

START WITH	CUBE CRUNCHER RULE	END WITH
1. 	Subtract 4 [P]	N N
2. 	Add 5 [N]	N N N
3. 	Subtract 3 [N]	P P P P
4. 	Add 6 [P]	P
5. 	Subtract 5 [P]	N N

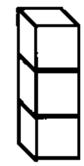
Quantity A	Quantity B	Independent Variable	As the independent variable increases, the dependent variable…
Age (yrs 1-10)	Height		
Flavor of gum	Time chewed		
Amount of payoff	Number of lottery winners		
Number of people in front of you in line	Time to wait in line		
Color of jeans	Number of times washed		
Number of bites taken	Length of banana		

FUNCTION MACHINES

Directions: *List the* output *of each of the following function machines. Complete the rule or draw the machine if not given.*

$y = x + 3$	$y = 2x$	$y = x^2$	$y = \frac{x}{2} - 1$
$x = 1, 2, 3, 4$	$x = 0, 2, 4, 6$	$x = 1, 2, 3, 4$	$x = 0, 2, 4, 6$
"Add 3"	"Double"	"Square"	"Divide by 2" "Subtract"
$y =$	$y =$	$y =$	$y =$
$y = 3x + 2$	$y = x^2 - 5$	$y = \lvert x \rvert$	$y = 4 + 2x$
$x = 1, 3, 5, 7$	$x = 1, 2, 3, 4$	$x = -2, -1, 0, 1$	$x = 3, 5, 7, 9$
"Multiply by 3" "Add 2"	"Square" "Subtract 5"	"Make Zero or Positive"	"Double" "Add 4"
$y =$	$y =$	$y =$	$y =$
$y =$	$y = 2x + 3$	$y =$	$y = \frac{x+3}{2}$
$x = 1, 2, 3, 4$	$x = 1, 2, 3, 4$	$x = 2, 4, 6, 8$	$x = 3, 6, 9, 12$
"Subtract 6"		"Multiply by 5" "Add 3"	
$y =$	$y =$	$y =$	$y =$
$y = \frac{x}{2} + 3$	$y =$	$y =$	$y = 3 - \frac{x}{2}$
$x = -4, -2, 0, 2$	$x = 1, 2, 3, 4$	$x = -3, -1, 1, 4$	$x = 0, 2, 4, 6$
	"Add 2" "Triple"	"Multiply by Itself"	
$y =$	$y =$	$y =$	$y =$

Activity 1: Painting Towers



Suppose you are painting a tower built from cubes, based on the pattern below. Use the table to find the relationship between the number of faces to paint and the number of blocks in the tower.
(Paint only the sides and the top.)

Term Number (Number of blocks)	Visual (Figure)	Written Description	Process Column	Numerical Value of Term (Faces to Paint)
1		A 1 cube-high tower has 5 faces to paint.		5
2				9
3				
4				
n				

1. Use the process column to write a function that expresses the relationship between the number of faces to be painted and the number of cubes.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range?

3. How many faces need to be painted for a 25 cube tower? Explain two ways of getting an answer.

4. If the tower you paint has 25 faces, how many cubes are in the tower? Explain two ways of getting an answer.

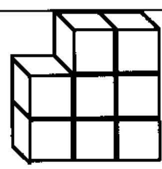
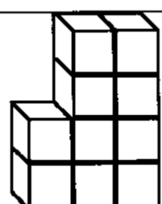
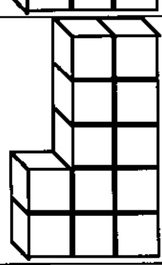
5. Suppose you have two adjacent columns of cubes instead of the one column as before. Use your cubes to build the first four figures and determine the number of faces that need to be painted.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term (Faces to Paint)
1				
2				
3				
4				
n				

6. Write a rule in words to describe how to find the total number of faces that need to be painted for two columns of cubes with 35 cubes in each column.
7. Write a rule in symbols that expresses the relationship between the number of cubes in each column and the total number of faces to be painted.
8. Predict how the graph of this data differs from the graph of the original data. Explain.
9. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph.

Activity 2: Building Chimneys

Suppose you are building a house with a chimney, based on the pattern below. Use the table to find the relationship between the number of blocks you need and the term number.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term (number of blocks)
1		A house with a chimney 1 block high takes 8 blocks to build.		8
2				10
3				
4				
n				

1. Use the process column to write a function that expresses the relationship between the total number of blocks needed to build the house and the term number.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range? Explain.

3. Use words to describe how to use blocks to build a house with a total of 52 blocks. What is this ordered pair on the graph?

4. If a house has a chimney that is 11 blocks high, how many blocks will you need to build the house? What is this ordered pair on the graph?

5. Does the ordered pair (13, 34) belong to this graph? How do you know?

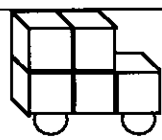
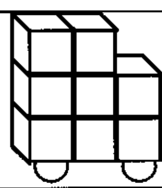
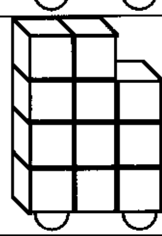
6. Suppose the chimney is made of 1 block instead of two and the house is built of three rows of 3 blocks instead of two rows of 3 blocks. Use your cubes to build the first three figures and record the data below.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term
1				
2				
3				
4				
n				

7. Write a rule for this new data that expresses the relationship between the total number of blocks and the number of blocks in the chimney for a house.
8. Predict how the graph of this data differs from the graph of the original data. Explain.
9. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph.

Activity 3: Constructing Trucks

Suppose you are building a truck, based on the pattern below. Use the table to find the relationship between the number of blocks you need and the figure number.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term
1		The truck has a base of 3 blocks with 2 blocks on top.		5
2				8
3				
4				
n				

1. Use the process column to write a function that expresses the relationship between the total number of blocks needed to build the truck and the term number.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range?

3. Find the total number of blocks needed for the 50^{th} figure.

4. If there are a total of 242 blocks, what term number is this?

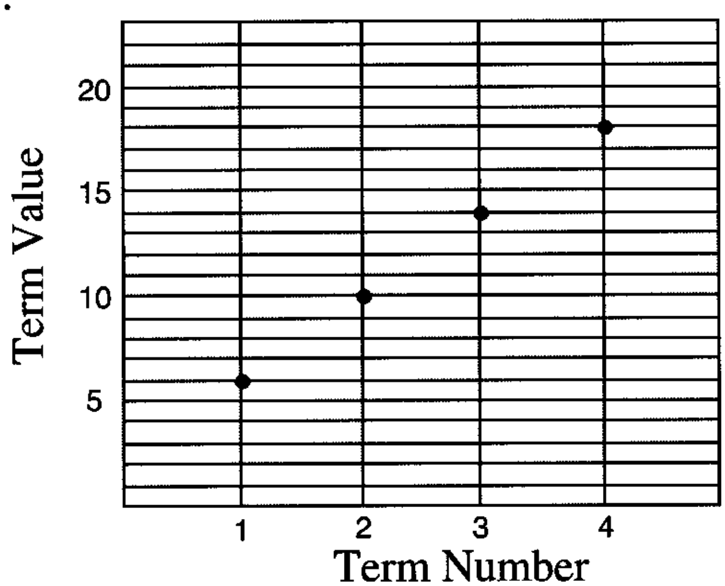
5. Suppose there is only one block on the top of each truck. Use your cubes to build the first three figures and record the data below.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term
1				
2				
3				
4				
n				

6. Write a rule for this new data that expresses the relationship between the total number of blocks and the figure/term number.
7. Predict how the graph of this data differs from the graph of the original data. Explain.
8. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph. What effect did changing the number of blocks on top of the truck have on the graph?
9. Suppose the original trucks (2 blocks on top) were built "double-wide." Predict how the graph differs from the original.
10. Build the first three "double-wide" trucks and graph the data on your graphing calculator. How does this graph compare to the graph of the original? Why?

Activity 4: Generating Patterns

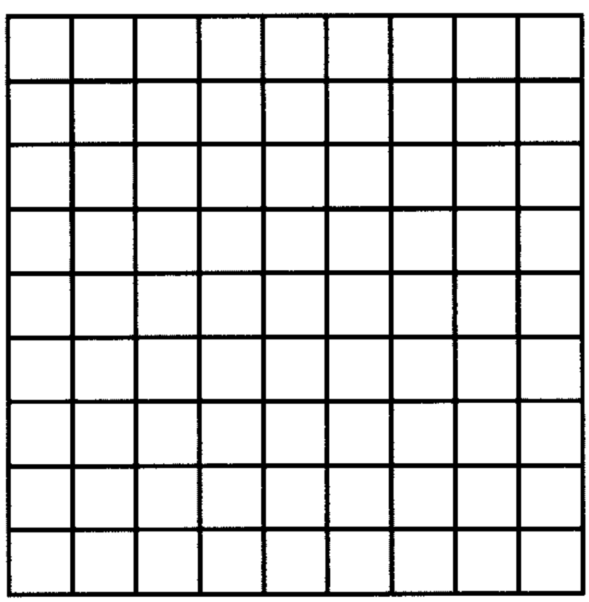
1. Given the following graph, use blocks to generate a sequence of figures that fits the data. Fill in the table and sketch the figures.



Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term
1				
2				
3				
n				

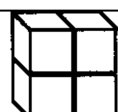
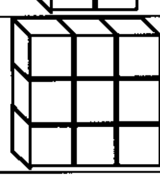
2. Given the function $y = 2x + 8$, use blocks to generate a sequence of figures that fits the function. Fill in the table, sketch the figures, and plot the graph. Label the graph.

Term Number	Visual (Figure)	Written Description	Process Column	Numerical Value of Term
1				
2				
3				
n				



Activity 1: Building Blocks

Suppose you are building square arrays out of cubes, as shown below. Use the table to find the relationship between the number of blocks needed for each figure and the dimension of the square array.

Term Number (dimension of square array)	Visual (figure)	Written Description	Process Column	Numerical Value of Term (number of blocks)
1		A 1 by 1 square has 1 cube.		1
2				4
3				
4				
n				

1. Use the process column to write a function that expresses the relationship between the term number and the number of cubes.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range?

3. What figure number will have 25 cubes? What is this ordered pair on the graph?

4. How many cubes do you need for the 25th term? What is this ordered pair on the graph?

5. Suppose you add an extra column of cubes to the figures above. Use your cubes to build the first four figures and record the data below.

Term Number	Visual	Written Description	Process Column	Numerical Value of Term
1				
2				
3				
4				
n				

6. Write a rule in words to describe how to find the total number of cubes for the fifth figure.

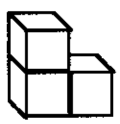
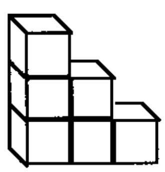
7. Write a rule in symbols that expresses the relationship between the total number of cubes and the term number.

8. Predict how the graph of this data differs from the graph of the original data. Explain.

9. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph.

Activity 2: Starting Staircases

Suppose you are building the triangular numbers from cubes, based on the pattern below. Use the table to find the relationship between the number of blocks and the term number.

Term Number	Visual	Written Description	Process Column	Numerical Value of Term
1		The 1st staircase needs 1 block.		1
2				3
3				
4				
n				

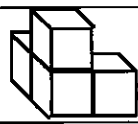
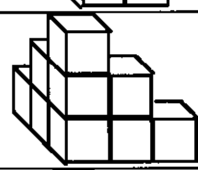
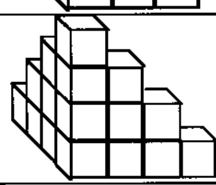
1. Use the process column to write a function that expresses the relationship between the total number of blocks and the term number.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range?

3. Use words to describe how to use blocks to build a similar staircase with a base of 15 blocks. What is this ordered pair on the graph?

4. If a similar staircase has 36 blocks, what is the figure number? What is this ordered pair on the graph?

5. Suppose the figures are as shown. Use your cubes to build the first three figures and record the data below.

Term Number	Visual	Written Description	Process Column	Numerical Value of Term (number of blocks)
1				
2				
3				
4				
n				

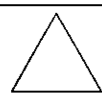
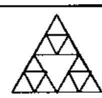
6. Write a rule for this new data that expresses the relationship between the total number of blocks and the term number.

7. Predict how the graph of this data differs from the graph of the original data. Explain.

8. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph.

Activity 3: Too Many Triangles

Suppose you are drawing the fractal, based on the pattern below. Use the table to find the relationship between the number of upward triangles and the term number.

Term Number	Visual	Written Description	Process Column	Numerical Value of Term (number of new upward triangles)
0		One new upward triangle		1
1		Three new upward triangles		3
2				
3				
4				
n				

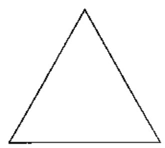
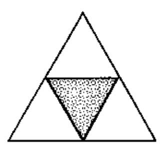
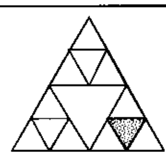
1. Use the process column to write a function that expresses the relationship between the number of new upward triangles and the term number.

2. Graph the data from your table on 1" graph paper and/or create a scatter plot on a graphing calculator. What is a reasonable domain for this situation? A reasonable range? Explain.

3. Find the total number of new triangles for term number 8.

4. If there are a total of 2187 new upward triangles, what term number is this?

5. Now consider the area of one of the smallest triangles in each figure and record the data below.

Term Number	Visual	Written Description	Process Column	Numerical Value of Term (area of smallest triangle)
0		The area of the smallest triangle is 1 unit		1
1		The area of one of the smallest triangles is $\frac{1}{4}$ of a unit		$\frac{1}{4}$
2				
3				
n				

6. Write a rule for this new data that expresses the relationship between the area of one of the smallest triangles and the term number.
7. Predict how the graph of this data differs from the graph of the original data. Explain.
8. Graph the above data on 1" graph paper and/or create a scatter plot on a graphing calculator and compare to the previous graph.

Movin' on Down the Line

Institute Notes

Concept: Use a graph to explore relationships between the quantities time and distance.

Overview: Participants actively collect distance data in relation to time and graph it. Through questioning and rich discussion, participants make the connection between the motions observed during the activity and the relationship shown in the graph. This activity requires extra physical space and can be done in a hallway or gym but gives the perfect opportunity to go outdoors on a day that weather permits.

TEKS Focus: **6.13**— The student uses logical reasoning to make conjectures and verify conclusions.
7.15— The student uses logical reasoning to make conjectures and verify conclusions.
8.16— The student uses logical reasoning to make conjectures and verify conclusions.

Also:
Grade 6
4A, 7, 8B, 10D, 12A, 13A
Grade 7
2G, 4B, 7A, 11B, 14A, 15A
Grade 8
4, 5A, 12B, 15A, 16A

Materials: Stop watches or watches with a second hand (one per group), 100' or 150' Measuring tapes (one per group), Data Collection Cards, Secret Instruction Cards, 1" grid wall charts, Peel-and-stick dots or markers

Procedure:

1. To prepare for this activity make a copy of the Data Collection Cards and Secret Instruction Cards sheets for each group and cut the cards apart. You may wish to laminate the cards.

2. Go over the directions on a transparency of Activity 1 - Instruction Sheet with the whole group. Do a quick demonstration and allow participants to ask questions about the procedure.

3. Divide the participants into groups of 12, distribute supplies, and have them complete Activity 1 by recording the distance data for the five movers.

4. Once the participants return to the room, have them complete Activity 1 by compiling their data into the tables and graphing the data on the grids.

Movin' on Down the Line

5. Have participants do Activity 2 by graphing the data on 1" grid wall charts. Label each graph with the name and number of the mover. You may wish to assign a mover (1 – 5) to each group so that the data for each mover is graphed only once.

6. When all graphs are completed, use the Reason and Communicate questions to discuss the information you can determine from the graphs.

7. Have participants work in their groups of twelve to discuss the graphs and determine the written instructions that each mover was given.

8. With the whole group, use a transparency of the Secret Instruction Card sheet to compare the instructions written by the groups to the actual secret instructions given to each mover.

9. Have participants reverse the process by doing Activity 2. Discuss using the Reason and Communicate questions.

10. Have participants use the rules they have formed to analyze the graphs in Activity 3. Discuss using the Reason and Communicate questions.

Extensions: Write a story about a person or some people moving. Draw a graph to illustrate it. The graph should show the relationship between the distances of the person or people from some fixed place at different times. Exchange graphs with another group. Write a story to go with the graph drawn by the other group. Compare your story with the other group's story.

Select a straight line section of one of the "Movin' on Down the Line" graphs and determine the speed of the mover and the direction in which he or she was moving. Repeat for several different sections. Write a brief description explaining how you determined the speed and direction.

Graph the data in both inches and feet and compare the graphs.

Movin' on Down the Line

Assessment: Illustrate the following situation with a graph that shows distance from the bus stop in relation to time:

Ann was walking toward the bus stop when she saw the bus coming. She ran as fast as she could toward the bus stop, but the bus left before she got there. She walked slowly the rest of the way to the stop and sat down to wait for the next bus.

Write a brief statement describing how you can determine the following when looking at a graph of distance from a fixed place over time.

- How can you tell when the mover is traveling fast, slow, or standing still?
- How can you tell if the mover is traveling toward or away from the beginning of the measuring tape?

Notes:

Movin' on Down the Line

Secret Instruction Cards

Movin' on Down the Line
Secret Instruction Card for Mover 1

Do not show this card to anyone.

Start at the beginning of the measuring tape. Walk forward slowly at a constant rate. When the timekeeper calls the 10^{th} second, stand still. When you hear 16, walk at a faster rate until time runs out.

Movin' on Down the Line
Secret Instruction Card for Mover 2

Do not show this card to anyone.

Start at the beginning of the measuring tape. Jog along the tape at a moderate speed for 10 seconds, keeping your speed as constant as you can. When the timekeeper calls the 10^{th} second, walk back toward the starting point at a slow, steady pace until time runs out.

Movin' on Down the Line
Secret Instruction Card for Mover 3

Do not show this card to anyone.

Start at the 25 foot mark of the measuring tape, facing the beginning of the measuring tape. Stand still for 6 seconds. When the timekeeper calls the 6^{th} second, walk along the tape toward the beginning of the tape at a constant speed. When you hear 12, stand still. When the timekeeper says 18, walk again until time runs out.

Movin' on Down the Line
Secret Instruction Card for Mover 4

Do not show this card to anyone.

Start at the far end of the measuring tape. Walk fast toward the beginning of the measuring tape. Keep your speed as constant as you can. When the timekeeper says 10, walk very slowly. Continue to walk very slowly toward the beginning of the tape.

Movin' on Down the Line
Secret Instruction Card for Mover 5

Do not show this card to anyone.

Start at the 10 foot mark of the measuring tape. Walk very, very slowly at first. Then very gradually increase your speed until you are running. When you reach the end of the tape, stop running and stand still.

Movin' on Down the Line

Data Collection Cards

Time: 0, 2 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Time: 4, 6 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Time: 8, 10 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Time: 12, 14 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Time: 16, 18 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Time: 20, 22 seconds	
Mover 1:	Position:
Mover 2:	Position:
Mover 3:	Position:
Mover 4:	Position:
Mover 5:	Position:

Movin' on Down the Line

Activity 1 - Instruction Sheet

Purpose: Collect data about the relationship between the distance of a person from the beginning of a measuring tape and the elapsed time.

In each group identify: 1 Timekeeper with a stopwatch or a watch with a second hand
5 Movers, each with a different Secret Instruction Card
6 Data collectors, each with a Data Collection Card

Description of the activity:
Each data collector will be given a Data Collection Card with two times written on the top. Each is responsible for recording the position of the mover at each time on his or her card. For example, if your card has 4 and 6 seconds on it, you are responsible for recording the position of each mover when the time-keeper calls 4 seconds and 6 seconds.

Each mover will be given a Secret Instruction Card. One at a time, follow the secret instructions on your card without showing others your card. The time-keeper will give the start signal and then call out each second for 22 seconds. ("Go,...1,...2,...")

First, mover 1 will practice moving down the line to give the data collectors the opportunity to position themselves where the mover will be at their assigned time. The second time Mover 1 moves down the line, each data collector will record the position of the BACK of the mover's back foot at his or her assigned time, rounding to the nearest foot.

Repeat this procedure for each mover.

Once all of the data is collected, return to the classroom.

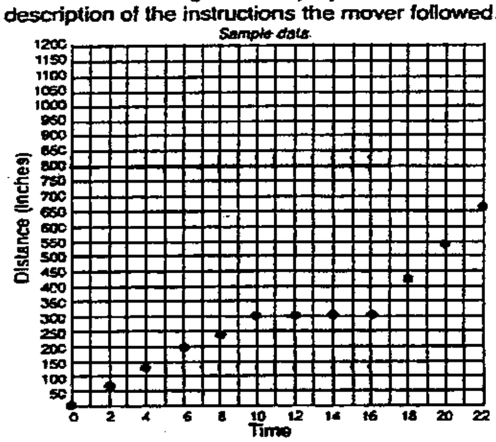
Movin' on Down the Line

Movin' on Down the Line

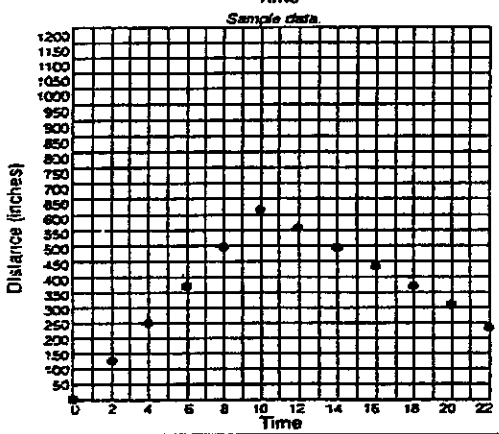
Activity 1, cont.

Enter the data for each mover into the tables below. Graph the data on the corresponding grids below and then transfer to the 1" grids to display on the wall. Based on each graph, write a description of the instructions the mover followed.

Mover 1	
Time	Distance
0	0
2	60
4	120
6	180
8	240
10	300
12	300
14	300
16	300
18	420
20	540
22	660



Mover 2	
Time	Distance
0	0
2	120
4	240
6	360
8	480
10	600
12	540
14	480
16	420
18	360
20	300
22	240



Math Notes:
In this activity, participants have recorded movers' distances at certain times, organized this data into a table, and then represented the data with a graph. From the data and graphs, they made conjectures about the movers' instructions. In the next activity, they will reverse the process. We call this habit of mathematical (algebraic) thinking **doing and undoing**. One of the processes we want to help develop during the institute and in our students is this notion of reversibility. That is, once we do something, often we can work backwards and gain new insight, a better understanding of what we have done, and a concrete knowledge of the concepts.

Also, one goal of this activity is for participants to form ideas about how the graph of a mover's movement looks for different instructions. In effect, we are building intuition for what motion looks like graphically as a relationship between distance and time. This is the first step toward developing algebraic reasoning by generalizing rules from data. We have not moved to a symbolic form with variables and equations yet, but we have begun to move from one representation to another. We call this habit of mathematical (algebraic) thinking **patterns to rules** (Driscoll).

Reason and Communicate:

- What does the ordered pair (x,y) mean in these graphs? *(time, distance)*

- In which graphs did the mover stand still for some period of time? How can you tell? *Graphs 1, 3, and 5. They have a horizontal line, which indicates that for more than one second the mover was at the same distance from the beginning of the tape.*

- In which graphs did the mover start at the beginning of the tape? *Graphs 1 and 2.*

- If the mover starts at the beginning of the tape, at what point does the graph start? *(0,0)*

- If the mover did not start at the beginning of the tape, where did he or she start? *In Graph 3, the mover started 25 feet from the beginning of the tape since the distance at t=0 is 25 feet (300 in.). In Graph 4, the mover started at the end of the tape since the distance from the beginning of the tape at t=0 is equal to the length of the tape. In Graph 5, the mover started 10 feet (120 in.) from the beginning of the tape since the distance at t=0 is 10 feet.*

- In which graphs is the mover moving away from the beginning of the tape? How can you tell? *In Graphs 1 and 5, the mover moves away from the beginning of the tape since the distance from the beginning of the tape is increasing over time. In Graph 2, the mover is moving away from the beginning of the tape at first.*

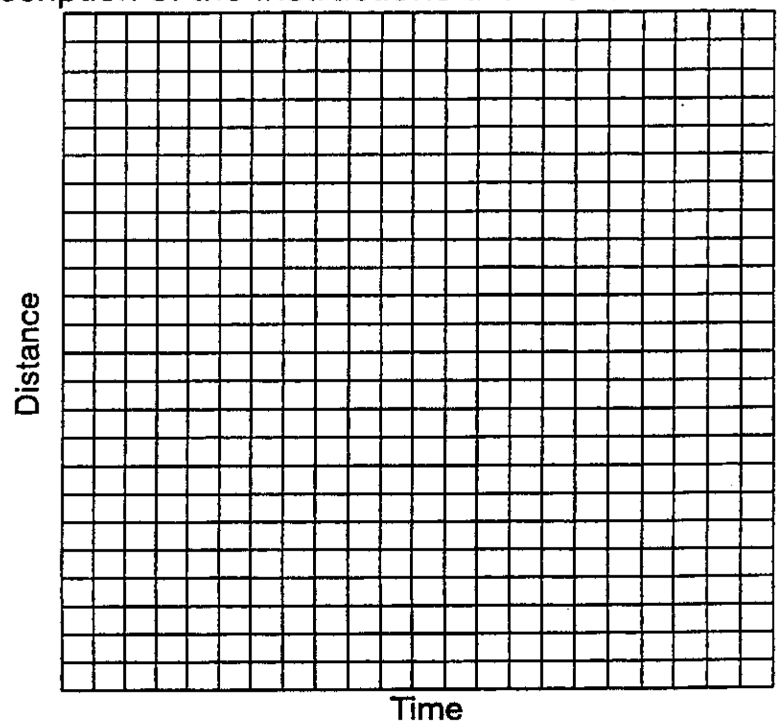
- In which graphs is the mover moving toward the beginning of the tape? How can you tell? *In Graphs 3 and 4, the mover is moving toward the beginning of the tape since the distance from the beginning of the tape is decreasing over time. In Graph 2, the mover moves toward the beginning of the tape in the second part.*

Movin' on Down the Line

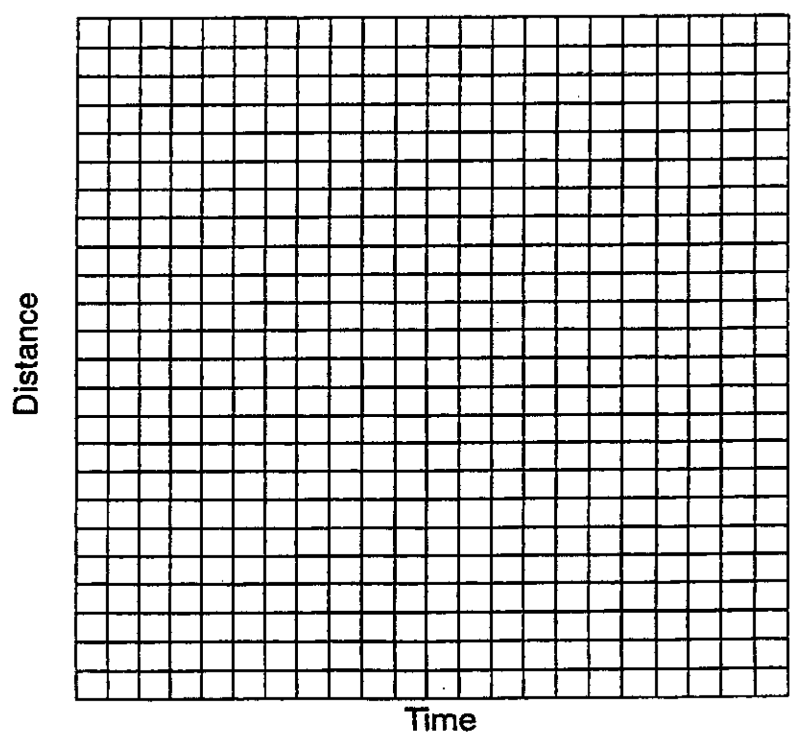
Activity 1, cont.

Enter the data for each mover into the tables below. Graph the data on the corresponding grids below and then transfer to the 1" grids to display on the wall. Based on each graph, write a description of the instructions the mover followed.

Mover 1	
Time	**Distance**



Mover 2	
Time	**Distance**

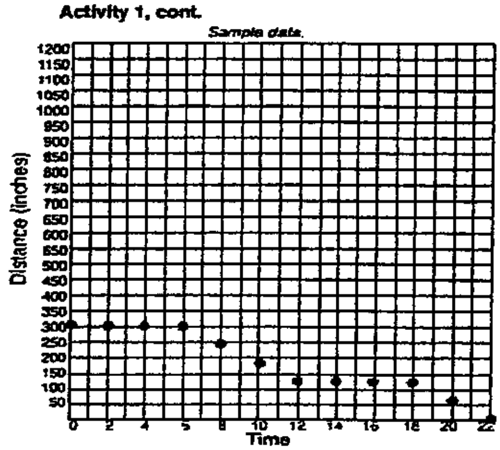


Movin' on Down the Line

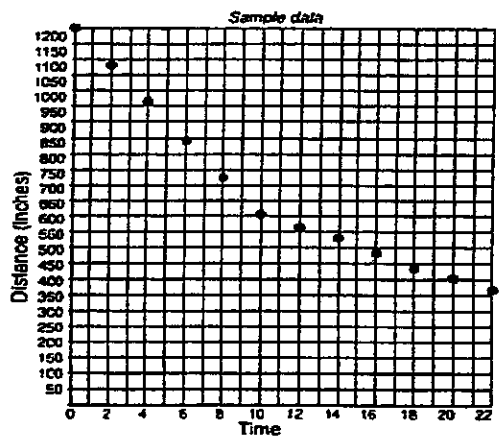
Movin' On Down the Line

Activity 1, cont.

Mover 3	
Time	Distance
0	300
2	300
4	300
6	300
8	240
10	180
12	120
14	120
16	120
18	120
20	60
22	0



Mover 4	
Time	Distance
0	1200
2	1080
4	960
6	840
8	720
10	600
12	560
14	520
16	480
18	440
20	400
22	360



Reason and Communicate, cont:

- What does the graph look like if the mover moves away from the beginning of the tape? *The graph rises from left to right.*

- What does the graph look like if the mover moves toward the beginning of the tape? *The graph falls from left to right.*

- In Graph 1, during which part is the mover moving the fastest? *The mover is moving fastest in the third section of the walk because then the change in his or her distance over one second is the greatest .*

- In Graph 2, when is the mover moving the slowest? *The mover is moving the slowest in the second section because the change in his or her distance over one second is the least then.*

- Comparing all five graphs, in which one is the mover moving the fastest? How can you tell? *Look for the steepest segment.*

- Comparing all five graphs, in which one is the mover moving the slowest? How can you tell? *Look for the most shallow segment.*

- Which graph does not look similar to the others? Why? *Graph 5 has a curved line instead of a straight line. The mover's rate is changing.*

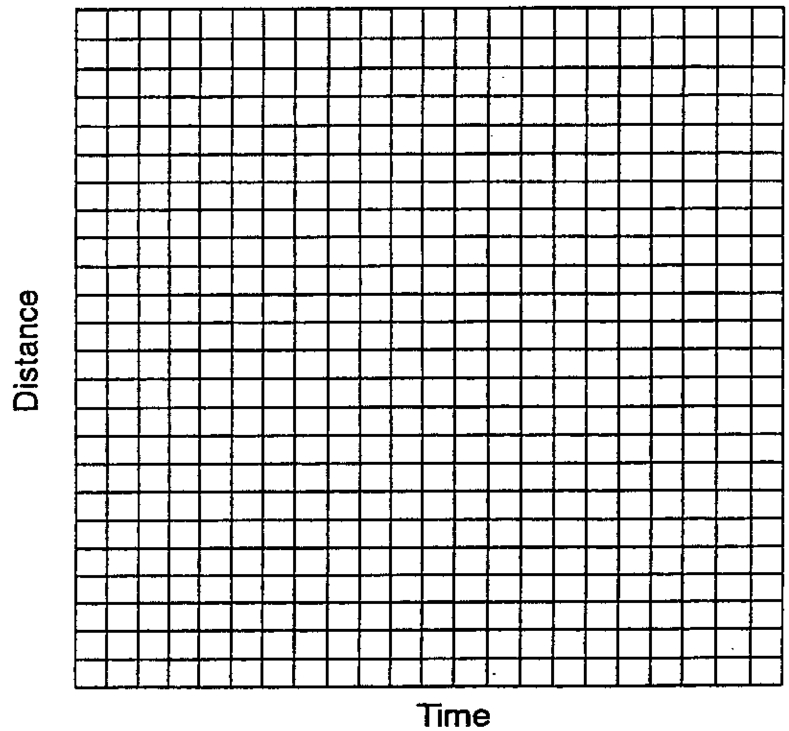
- Should you connect the data points? Is the data discrete or continuous? *Discuss that the only way you would have a break in the graph is if the mover left the course and then came back.*

- The graph represents a relationship between what two quantities? *Elapsed time and distance from the beginning of the measuring tape.*

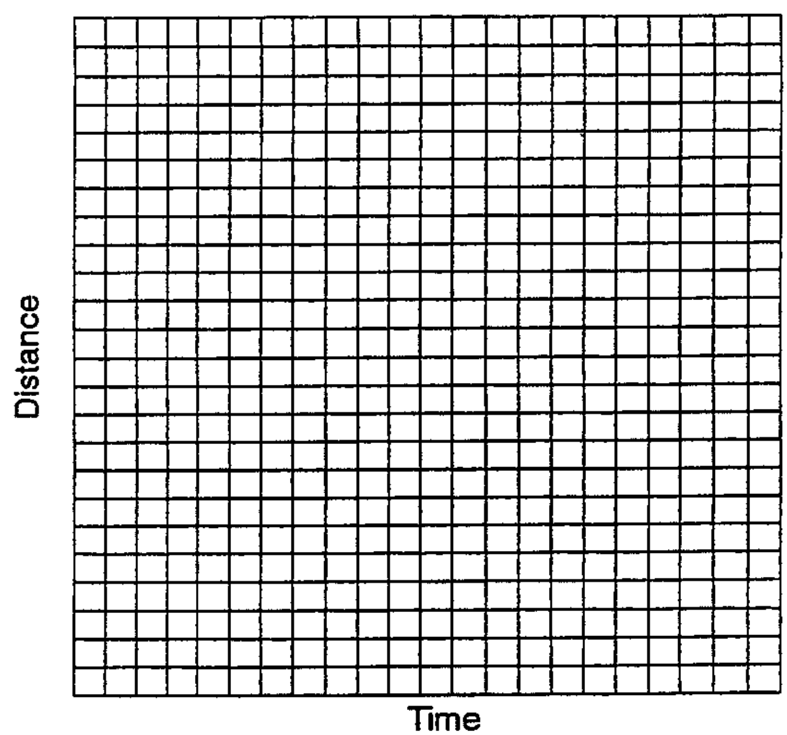
Movin' on Down the Line

Activity 1, cont.

Mover 3	
Time	**Distance**



Mover 4	
Time	**Distance**

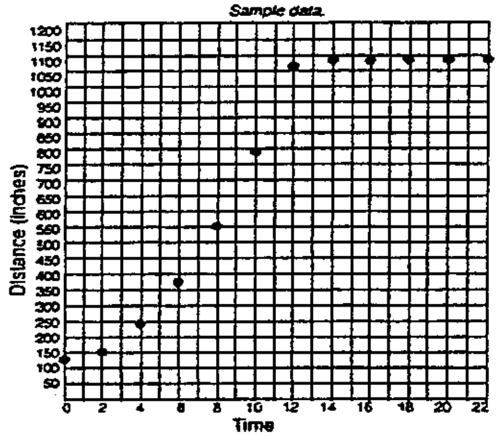


Movin' on Down the Line

Movin' On Down the Line

Activity 1, cont.

Mover 5	
Time	Distance
0	120
2	144
4	228
6	360
8	540
10	788
12	1044
14	1200
16	1200
18	1200
20	1200
22	1200

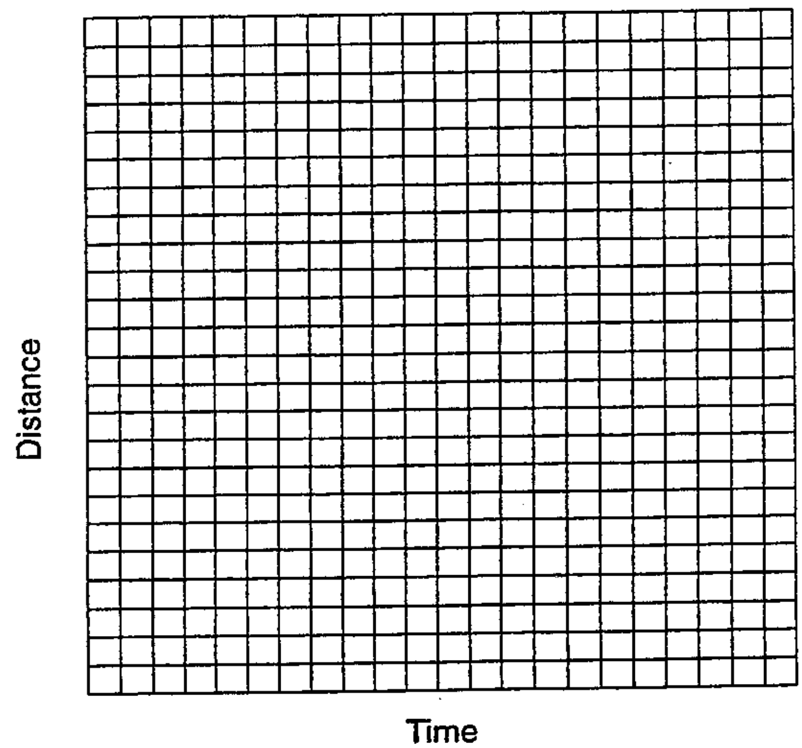


TEXTEAMS Rethinking Middle School Mathematics: Algebraic Reasoning Activity-13

Movin' on Down the Line

Activity 1, cont.

Mover 5	
Time	**Distance**



Movin' on Down the Line

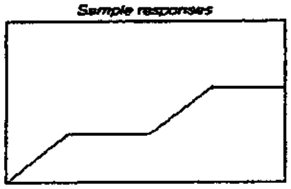
Movin' On Down the Line

Activity 2

Some students in another class were given different Secret Instruction Cards for "Movin' on Down the Line." Data was collected for 20 seconds. Below is a list of what was on their cards. Draw a sketch of what you think the graph should look like for each one.

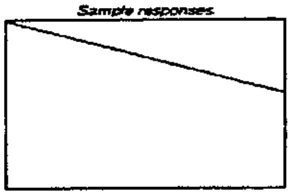
1. Start at the beginning of the measuring tape. Walk forward at a steady, slow pace for 5 seconds. Then stand still for the next 5 seconds. Repeat these steps until time runs out.

Sample responses



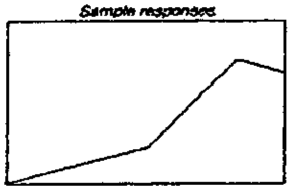
2. Start at the end of the tape and walk towards the beginning of the tape at a steady pace.

Sample responses



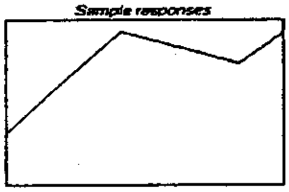
3. Start at the beginning of the measuring tape and walk slowly for 8 seconds then run for 8 seconds. Then turn around and walk slowly back to the beginning of the tape.

Sample responses



4. Start at the 50 foot mark on the measuring tape. Walk forward quickly for 7 seconds. Then turn around and walk slowly back toward the beginning of the tape for 7 seconds. Turn around again and walk away at a medium rate from the beginning of the tape until time runs out.

Sample responses



TEXTEAMS Rethinking Middle School Mathematics: Algebraic Reasoning Activity-15

Math Notes:
Point out that in Activity 1 participants recorded movers' distances at certain times, organized the data into a table, and represented the data on a graph. From the data and graphs, we made conjectures about the movers' instructions. Now in Activity 2, we have reversed the process. We are **undoing** by taking the verbal description of a mover's walk and representing it with a graph.

Also, we used the algebraic thinking skill of **patterns to rules** by applying the rules we found in Activity 1. For example, a faster pace is represented by a steeper line.

Reason and Communicate:

- How are the graphs alike? How are they different?

- Which graph has a horizontal section? *Graph 1*

- How do you represent on the graph a mover standing still? Explain why. *By drawing a horizontal line to show that the distance is staying the same as the time changes.*

- Which graph slopes down from left to right? Why does it slope that way? *Graph 2 slopes down from left to right because the mover started at the end and walked toward the beginning. Thus, the distance from the beginning of the tape decreased in relation to time. Also, Graphs 3 and 4 had segments where the mover turned around and walked toward the beginning, so they have segments that slope down from left to right.*

- Which graph has the steepest segment? Why? *Graph 3, because the mover ran and there was a greater change in distance per second.*

- How did you represent a mover walking at a slow pace? Why? *By drawing a shallow line to show a small change in distance for each second of time.*

- Does it matter how shallow you draw the line? *At this point, we are not overly concerned with finding specific rates of change. We just want participants to draw things slow and fast relative to one another.*

Movin' on Down the Line

Activity 2

Some students in another class were given different Secret Instruction Cards for "Movin' on Down the Line." Data was collected for 20 seconds. Below is a list of what was on their cards. Draw a sketch of what you think the graph should look like for each one.

1. Start at the beginning of the measuring tape. Walk forward at a steady, slow pace for 5 seconds. Then stand still for the next 5 seconds. Repeat these steps until time runs out.

2. Start at the end of the tape and walk towards the beginning of the tape at a steady pace.

3. Start at the beginning of the measuring tape and walk slowly for 8 seconds then run for 8 seconds. Then turn around and walk slowly back to the beginning of the tape.

4. Start at the 50 foot mark on the measuring tape. Walk forward quickly for 7 seconds. Then turn around and walk slowly back toward the beginning of the tape for 7 seconds. Turn around again and walk away at a medium rate from the beginning of the tape until time runs out.

Movin' on Down the Line

Movin' On Down the Line

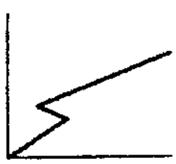
Activity 3

Study the graphs below made by a group of students in another class. Are these graphs possible graphs for other Secret Instruction Cards for "Movin' on Down the Line"? If not, explain why. If possible, write a description of how the person was moving.

1.

2.

3.



TEXTEAMS Rethinking Middle School Mathematics: Algebraic Reasoning Activity-17

Answers:

1. Start about in the middle of the tape measure, walk further away for a short while, and then stand still for the remainder of the time.

2. Start at the beginning of the tape measure and stand still for a fourth of the time. Then walk away from the beginning of the tape measure until time runs out.

3. This graph is not possible for this activity. At one time, the mover is three different distances from the beginning of the tape. This implies that the walker was in three places at once. Note that Exercises 1 and 2 are functions and Exercise 3 is not a function.

Math Notes:
Remember that we are looking at the relationship between the two quantities time and distance from the beginning of the tape. The graphs represent these relationships. The graphs do not represent a picture of the path taken. The mover did not walk a course that looks like the graph.

In Activity 2, we went from a description to a graph. In Activity 3, we move from a graph to a description. This idea of going back and forth between representations, **doing and undoing**, is a major mathematical habit of thinking that builds algebraic reasoning. Keep looking for it throughout the institute.

In Activity 3, we build intuition for the algebraic concept of function. Algebra students will need to differentiate between relations that are functions and relations that are not functions.

Reason and Communicate:

- Which graphs are possible graphs for Secret Instruction Cards? Why? *Graphs 1 and 2 show that each moment the mover is only at one distance from the beginning of the measuring tape.*

- Which graph is not a possible graph of distance from the beginning of the measuring tape in relation to elapsed time? Why? *Graph 3 is not possible because to produce this graph, the mover would have to be in up to three places at the same moment in time.*

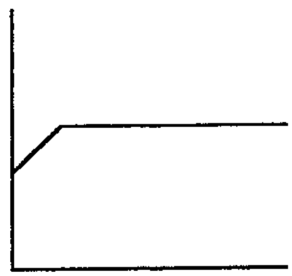
- What do we mean by **doing and undoing**? *Doing and undoing is a mathematical (algebraic) habit of mind. It is the idea of reversibility, strengthening of mathematical connections by going backwards. Activity 2 reverses Activity 1. Activity 3 reverses Activities 1 and 2.*

Movin' on Down the Line

Activity 3

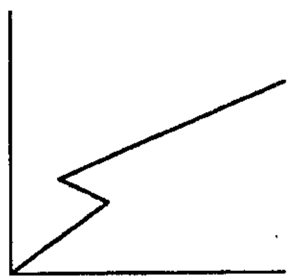
Study the graphs below made by a group of students in another class. Are these graphs possible graphs for other Secret Instruction Cards for "Movin' on Down the Line"? If not, explain why. If possible, write a description of how the person was moving.

1.



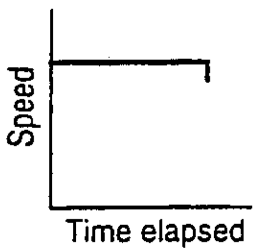
2.

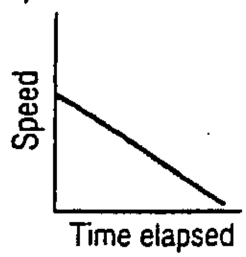
3.

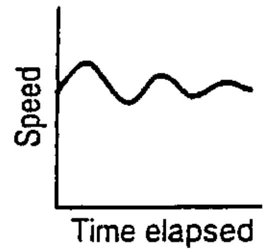


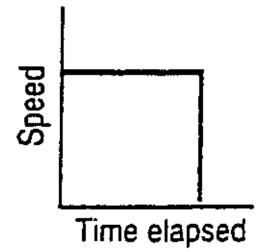
Indicate which graph matches the statement.

1. A train pulls into a station and lets off its passengers.

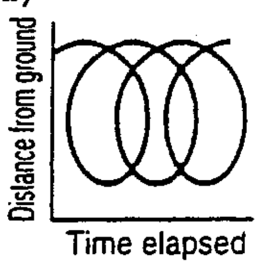
a)


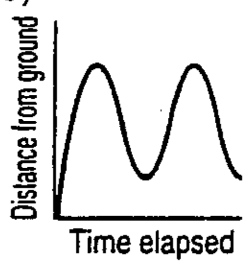
b)


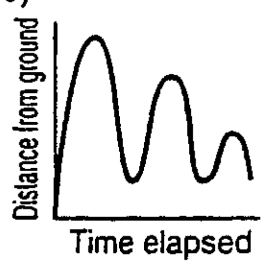
c)


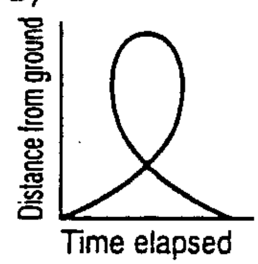
d)


2. A man takes a ride on a ferris wheel.

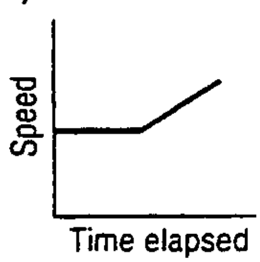
a)


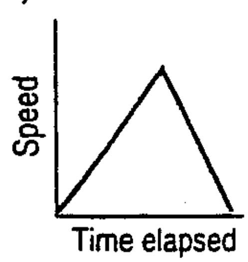
b)


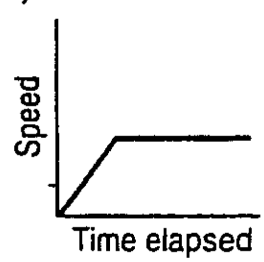
c)


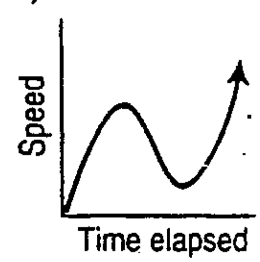
d)


3. A woman climbs a hill at a steady pace and then starts to run down one side.

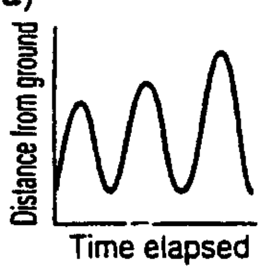
a)


b)


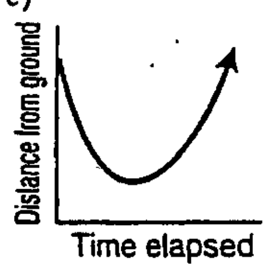
c)


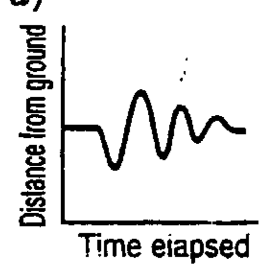
d)


4. A child swings on a swing.

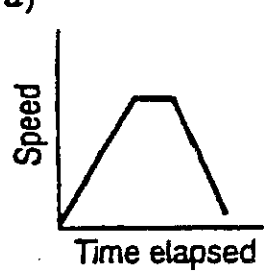
a)


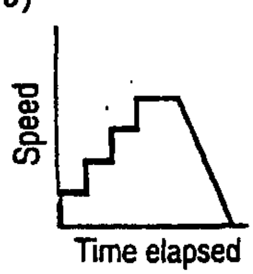
b)

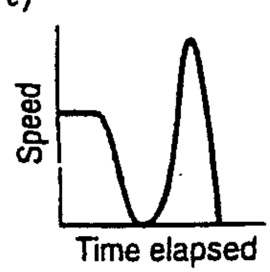

c)


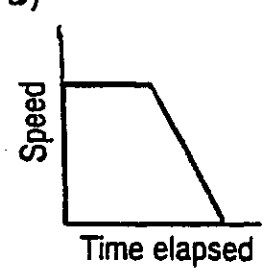
d)


5. A child climbs up a slide and then slides down.

a)


b)


c)


d)


From the *Mathematics Teacher*, September 1994

WHAT'S MY RULE?

When solving problems, it is often helpful to know a general rule. Finding the *n*th term in a sequence is finding the general rule for all the terms in that sequence.

Complete each sequence and find the *n*th term. *n*th term

1 2, 4, 6, 8, 10, ___, ___, ___, . . . _____

2 1, 3, 5, 7, 9, ___, ___, ___, . . . _____

3 3, 6, 9, 12, 15, ___, ___, ___, . . . _____

4 7, 10, 13, 16, 19, ___, ___, ___, . . . _____

5 5, 10, 15, 20, 25, ___, ___, ___, . . . _____

6 2, 7, 12, 17, 22, ___, ___, ___, . . . _____

7 $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, ___, ___, ___, . . . _____

8 $-\frac{3}{2}$, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, ___, ___, ___, . . . _____

9 17, 25, 33, 41, 49, ___, ___, ___, . . . _____

10 9, 23, 37, 51, 65, ___, ___, ___, . . . _____

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FOLLOW THE ORDERS !

These numbers in ordered pairs are related to each other. Discover what holds these pairs together and fill in the blanks.

1. (1, 2), (3, 4), (5, 6), (0, ___), (10, 11), (2, ___), (___ , 7), (8, ___)
2. (2, 4), (6, 12), (___ , 6), (4, ___), (13, ___), (___ , 10)
3. (9, 3), (3, 1), (12, ___), (___ , 2), (15, ___), (___ , 8)
4. (1, 3), (7, 9), (3, 5), (8, ___), (16, ___), (___ , 13), (6, ___)
5. (4, 0), (2, 0), (9, 0), (12, ___), (6, ___), (22, ___)
6. (4, 16), (7, 49), (5, ___), (___ , 64), (10, ___), (___ , 36)
7. (4, 9), (5, 11), (2, 5), (6, ___), (___ , 7), (10, ___), (___ , 3)
8. (10, 2), (25, 5), (5, 1), (35, ___), (50, ___), (___ , 9), (___ , 3)
9. (24, 3), (16, 2), (40, 5), (32, ___), (___ , 7), (64, ___)
10. (3, 10), (6, 37), (2, 5), (7, ___), (5, ___), (___ , 2), (___ , 82)
11. (88, 8), (44, 4), (55, ___), (11, ___), (___ , 6), (___ , 10)
12. (12, 8), (6, 5), (10, 7), (14, ___), (20, ___), (___ , 14), (100, ___)

EXTRA CHALLENGES:

1. (3, 6), (8, 56), (5, 20), (6, ___), (10, ___), (4, ___)
2. (16, 2), (64, 4), (36, 3), (100, ___), (4, ___), (144, ___)

Activity 2: Guess My Function!

Write the symbolic rule below each table:

1.

x	y
0	11
1	16
2	21

2.

x	y
0	1.3
1	2.5
2	3.7

3.

x	y
0	4
1	2
2	0

4.

x	y
0	25
1	18
2	11

5.

x	y
1	20
2	24
3	28

6.

x	y
5	10.1
6	13.1
7	16.1

7.

x	y
3	21
4	20
5	19

8.

x	y
10	-9
11	-14
12	-19

9.

x	y
0	11
2	23
4	35

10.

x	y
0	15
5	10
10	5

11.

x	y
0	4
3	10
6	16

12.

x	y
0	25
10	20
20	15

13.

x	y
2	10
5	16
8	22

14.

x	y
11	5.3
15	4.3
19	3.3

15.

x	y
-7	14
-5	10
-3	6

16.

x	y
-10	0
-4	3
2	6

Activity 3: Finite Differences

When only one level of differences is necessary to obtain a constant value, the algebraic rule which generates the terms of the sequence is linear and can be written in the form $b + an$. The terms of a linear sequence are in the form b, $b+1a$, $b+2a$, $b+3a$, $b+4a$, . . . , $an+b$.

Δx	Term #	Process Column	Value of Term	Δy
	0	____________	b	
___				___
	1	____________	$b+1a$	
___				___
	2	____________	$b+2a$	
___				___
	3	____________	$b+3a$	
___				___
	4	____________	$b+4a$	
	n	$a(n)+b$	$an+b$	

1. If first differences are constant, then ____________________

2. If the data is linear, then ______________________________

Reading the Graph

Updated: 08/21/08

Objectives:
Students will interpret, analyze and formulate conclusions based on a graph of two different functions. They will formulate an equation of a line using function notation, find the average rate of change over a specific time interval, find values from the graph, interpret the meanings of the problem situation, and interpret a graph with a verbal description.

Connections to Previous Learning:
Students should be able to determine average rate of change and write verbal descriptions of graphs.

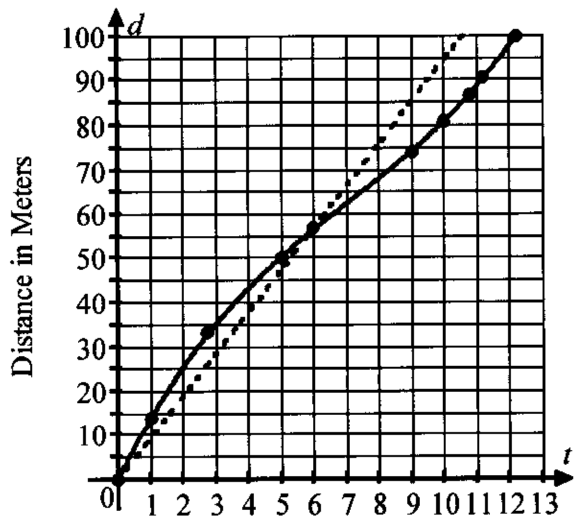
Connections to AP*:
AP Calculus Topics: Rate of Change; Position/Velocity/Acceleration

Time Frame:
45 minutes

Materials:
Student activity pages

Teacher Notes:
This lesson can be used as an introduction to average rate of change and as such should be worked through as a class activity. It can also be used as a review or as an assessment.

Reading the Graph



The graph above compares the position for two runners in the women's 100-meter dash in two different Olympics. Both women won the gold medal in their respective Olympics. The position at time, *t*, for Betty Robinson is represented by the solid line, and the position of Florence Griffith-Joyner at time, *t*, is represented by the dotted line.

1. a) State the independent variable.

   b) State the dependent variable.

2. a) Both ladies won the gold medal. Who do you think won the gold medal (1st place) in 1928 and who won the gold medal in 1988?

   b) Explain the reasoning behind your choice of winners in part (a).

   c) What was approximate winning time in 1988?

   d) Approximately how many seconds were shaved off of the winning time in these 60 years?

3. a) What was Robinson's average speed for the entire race?

   b) What was Griffith-Joyner' average speed for the entire race?

4. a) If the runners had run the race together, at what time would one runner pass the other? Round the answer to the nearest second.

   b) Approximately how far along the course were Griffith-Joyner and Robinson when this happened?

   c) Who would have overtaken whom to win the race?

5. a) Explain the meaning of the statement, $d(t) = 70$.

   b) If $d(t) = 70$, approximate t for each runner.

   c) Explain meaning of $d(10)$.

   d) Approximate $d(10)$ for each runner.

6. a) How does Robinson's speed over the interval from 0 to 2 seconds compare with Griffith-Joyner's speed over the same interval? Explain your answer using the graph.

   b) Find Griffith-Joyner' average speed over the interval from 5 to 9 seconds.

   c) Find Robinson's average speed over the interval from 5 to 9 seconds.

   d) What do the values in parts (b) and (c) represent? Use the units of measurement.

7. Suppose the ladies are running on the same track at the same time and that you are the commentator for the race. Write a paragraph describing the race that both ladies ran. Indicate in your narrative when or if they increased or decreased their speed. Indicate their positions in comparison to each other. In addition, be sure to speculate what Robinson was planning at about 5 seconds.

Reading the Graph

Answers:

Students answer will vary when values are read from the graph.

1. a) time in seconds
   b) distance in meters

2. a) In 1928 Robinson won and in 1988 Griffith-Joyner won.

   b) Over the years, the women have gotten faster and stronger with weight training so Griffith-Joyner finished in a shorter amount of time.

   c) Approximately 10.5 seconds (actually 10.54 seconds, an Olympic record).

   d) About 1.5 seconds (actually 1.66 seconds).

3. a) $\frac{100}{12.2}$ or 8.197 meters per second

   b) Approximately $\frac{100}{10.5}$ or 9.524 meters per second (actually $\frac{100}{10.54}$ or 9.488 meters per second)

4. a) 6 seconds

   b) About 57 meters

   c) Griffith-Joyner overtook Robinson.

5. a) At time t, the runner is at the 70 meter mark.

   b) Robinson's time is approximately 8.4 seconds and Griffith-Joyner's time is about 7.4 seconds.

   c) The runner's distance from the starting point at 10 seconds

   d) Exactly 10 seconds after the race started, Robinson had run approximately 80 meters and Griffith-Joyner had run approximately 95 meters.

6. a) For the first two seconds of the race Griffith-Joyner ran at a constant rate of about 9.488 meters per second while Robinson ran at an average rate of approximately 12.5 meters per second. If they had been running on the same track, Robinson would be ahead after 2 seconds by approximately 5 meters.

   b) About 9.488 meters per second (This can be calculated because her speed is constant)

   c) 6.25 meters per second

   d) These values would be the speed, in meters per second, of the runners if each were running at a constant speed between 5 and 9 seconds. They are the average speeds of each runner.

7. Both the runners begin the race at the same time with Robinson pulling ahead and running at a faster speed for about the first 3 seconds. Robinson slows down, and Griffith-Joyner passes her at about 6 seconds. Robinson now begins to run faster, but she cannot catch Griffith-Joyner. It seems as if Robinson has run out of steam too early ,and Griffith-Joyner finishes the 100 meter race at about 10.54 seconds (the Olympic record) and Robinson comes in at a time of 12.2 seconds.

Average Rates of Change

Example:
On his fifth birthday, Paul was 42 inches tall. On his seventh birthday, he was 48 inches tall.

Find the average rate of change. Write a sentence interpreting the meaning of this rate of change.

At 3 o'clock, Sharon passes mile marker 295 on Highway 35. At 6 o'clock she passes mile marker 475.

Dixie left Austin with a full tank of gas (16 gallons) and an odometer reading of 12,584 miles. Upon arriving in Houston, her gas tank was only half full and her odometer reading was 12,792.

Scott began printing his history paper at 3:15. At 3:20, he found that it had printed 12 of his 15 pages.

The value of Robert's new car after 2 years was $11,200. When the car is 6 years old, the value has dropped to $6100.

Name_______________________________________ Class Period ____________

Distance Match

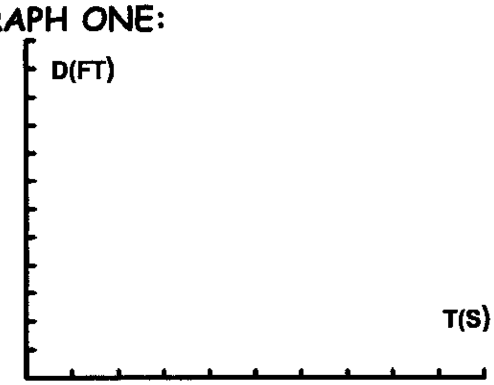
Follow these directions to set up your calculator-CBR unit.

Directions for CBR1
1. Press APPS
2. Choose CBL/CBR
3. Press ENTER
4. Choose RANGER
5. Press ENTER
6. Choose APPLICATION
7. Choose FEET
8. Choose DIST MATCH
9. Press ENTER

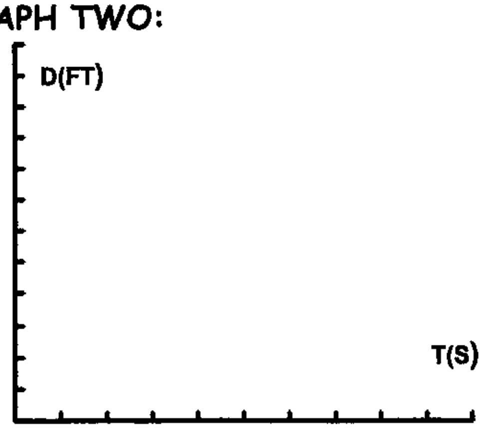
Directions for CBR2
1. Press APPS
2. Choose EasyData
3. Choose SetUp
4. Choose Distance
5. Choose UNITS
6. Choose (ft)
7. Choose OK
8. Choose SetUp
9. Choose Distance Match
10. Choose Start

The Activity:
1. Record the graph that you see on one of the blank graphs below.
2. Practice walking to match the graph. When you feel like you've matched it well, write a description of your movement that matched the graph. Use the words "time" and "distance" in your descriptions.
3. Repeat steps 1 and 2 with as many different graphs as you can in the allotted time.

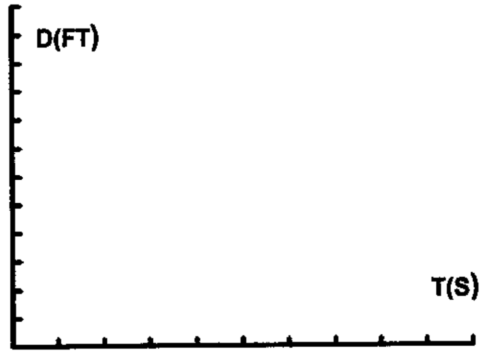
GRAPH ONE:



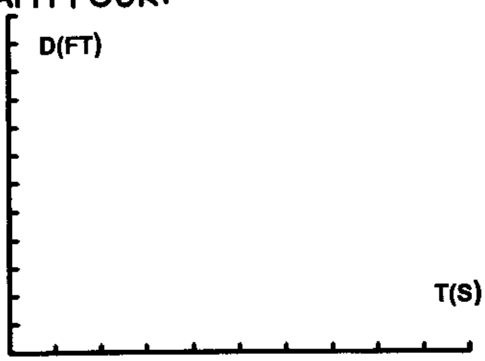
GRAPH TWO:



GRAPH THREE:

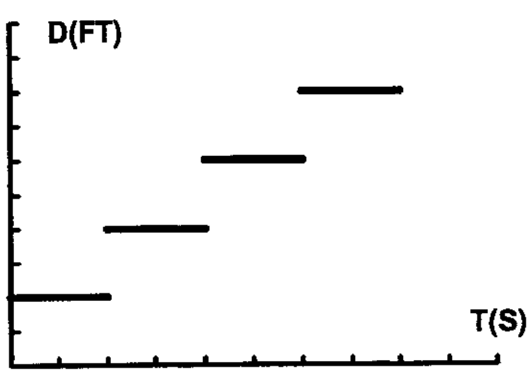


GRAPH FOUR:



Extension:

As a challenge, see if you can figure out a way to obtain the following graph. Write a description of the steps you would take. Be prepared to show the class how you matched the graph.



Slope, Slope Baby

Master Rapper Kitty "Cool" Morgan

Yo Vip, Let's kick it

Slope, Slope Baby
Slope, Slope Baby

I am the rate of change
To some of you that may sound strange

I am the change in y over x
I'm how you get from one point to the next

I'm constant, but only for a line
For all functions I'm changing all the time

Sometimes I'm called rise over run
It may sound simple but I'm a lot of fun

I am what gives the line its lean
They call me m cause I'm so mean

To graph a line
don't you start with me

Start with my friend
His name is b

He's an intercept
But plot him on the y

He's all by himself
But he's a great guy

From there start
with the number on the top

Go up that many
If it's negative then drop

Now it's time
to go left or right

The number on the bottom
you need in sight

Once you've done
Your rise over run

Plot that point
I told I'm fun

Repeat now
and get point 3

Connect them and
You have a line you see

Slope Slope Baby

I am constant
when you graph a line

Rate of Change Baby

Slope, Slope Baby

Yo I'm Outta Here
Word to your mother

Homewood H
1901 South Lake
Homewood,
(205)
Kmorgan@homewoo
KITTY MORGAN
Alabama Secondary Teacher of the Yea
2002-2003

Number Patterns

Concept Areas

Sequences of numbers. Finding the next number in the pattern. Students also use algebra to relate the numbers to one another.

The problem *Feeding Frenzy* (page 137) has an attribute common to several of the more advanced problems: there is more than one solution without the optional clue; the optional clues make the solution unique. Chances are students will come up with the unique solution anyway, but in case they don't, be prepared with a good question or two.

For Each Group:

- Manipulatives.
- Pencil and paper.

Description

Each problem has a sequence of numbers. Groups need to figure the sequence out. In some problems, the clues relate members of the sequence to one another ("The third number is twice the second number plus one."). You might think of solving the problem as if it were algebra, and then trying to figure out the pattern based on the results. Other problems are more situational in nature, depicting a situation where a number pattern arises naturally.

Whichever way the problem is set up, these are multiple-step problems ideal for groups. As with the *School Math* problems (page 76), the group has to hold a lot of different ideas simultaneously.

Features

Once the numbers in the sequence are established, the group still has to figure out its rule! Extending sequences is a classic problem that appears in all sorts of math tests. More important, extending sequences can help students deepen their understanding of functions. The idea of a function is central to mathematics from algebra on and is vital to success in a math-based career.

Possible Debriefing Questions

Which patterns were harder to figure out?

Which was easier—finding the numbers in the patterns or figuring out how to extend the patterns?

What sort of strategies did your group use to extend the patterns?

If you extend a pattern, how do you know if you are right?

Pattern-o-Piles

The second pile has one-fifth as many as the fifth pile.

There is only one in the first pile.

If the pattern continues, how many are in the sixth pile?

Pattern-o-Piles

The second pile has half as many as the third pile.

If the pattern continues, how many are in the sixth pile?

Pattern-o-Piles

The fourth pile is one third less than the fifth pile.

If the pattern continues, how many are in the sixth pile?

Pattern-o-Piles

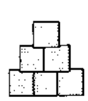
If you put the first, second and third pile together, you would get the fourth pile.

If the pattern continues, how many are in the sixth pile?

Pattern-o-Piles

If the piles were people, each pile would have a good number for making a human pyramid. (One person is a small pyramid.)

If the pattern continues, how many are in the sixth pile?



Pattern-o-Piles

The fourth pile is the third pile plus four.

If the pattern continues, how many are in the sixth pile?

from *Get It Together* • EQUALS, Lawrence Hall of Science

What's the Pattern?

The sixth number is the third number times four and it is the first number times eight.

What is the seventh number in the pattern?

What's the Pattern?

The third number is the second number plus one, and the fourth number is the third number plus one.

What is the seventh number in the pattern?

What's the Pattern?

The fifth number is the third number plus the fourth number.

What is the seventh number in the pattern?

What's the Pattern?

When you add the first six numbers together, the sum is twenty.

What is the seventh number in the pattern?

What's the Pattern?

The third number raised to the third power equals the sixth number.

What is the seventh number in the pattern?

What's the Pattern?

The first and second numbers are the same.

What is the seventh number in the pattern?

Annabelle Arable

Annabelle Arable was a successful farmer and landowner. She started out with only ten acres of land in the year after the Big Drought.

What year was the Big Drought?

Annabelle Arable

Annabelle Arable was so successful that after every fall harvest, she bought all the fields that shared a fence with her own.

Those other farmers left Cakewalk County for the Big City to seek their fortunes.

Annabelle Arable

In Cakewalk County, where Annabelle lived, each field is perfectly square and shares a fence with the four fields that surround it. Each field is also exactly ten acres.

Annabelle Arable

In the summer of 1914, Annabelle had 410 acres of land.

When was the Big Drought in Cakewalk County?

Annabelle Arable

Every year, Annabelle Arable's total landholdings were in the shape of a square (sort of), though she never held a square number of acres or fields.

By the way, there are 640 acres in a square mile.

Annabelle Arable

In May 1915, at the age of seventy-three, Annabelle retired.

She gave 200 acres to each of her three children and kept ten for herself, where she raised prizewinning asparagus for many years.

The Bus Stops Here

A city bus with forty-five seats picks up its first passenger at First Street.

The Bus Stops Here

The bus picks up its second passenger at Second Street.

Once all the seats are filled, people have to stand. In what block do people start having to stand?

The Bus Stops Here

Starting with Third Street (the third stop), the number of people who get on the bus is equal to the sum of the boarders from the previous two stops.

The Bus Stops Here

Everyone who gets on the bus gets off three blocks after they get on.

The Bus Stops Here

If everyone got off after two blocks instead of three, the number of riders when the bus pulled up would be the same as the number of people waiting for the bus at each stop.

The Bus Stops Here

There are six people in the bus between Fourth and Fifth streets.

Two people get off at Sixth Street.

Feeding Frenzy

Bugwumps are usually very sweet, but every two months, in the middle of the night, they go on a feeding frenzy and devour trundles as quickly as possible.

On May 1, 1934, there were thirty-six Trundles.

Feeding Frenzy

During the first month after a feeding frenzy, the Trundles quadruple in population.

Here's the question: How many Trundles were there on the morning of January 1?

Feeding Frenzy

During the night the Bugwumps feed, they gobble up five out of every eight Trundles.

This problem takes place between January 1 and May 1 of the year 1934.

Feeding Frenzy

During the second month following a feeding frenzy, the population of Trundles doubles.

The Bugwump feeding frenzies always occur during the night preceding the first day of a month.

Feeding Frenzy

The Bugwumps ate twenty Trundles during the frenzy on the night of February 28, 1934.

No one had ever seen a Bugwump (or a Trundle, for that matter) until late December, 1933.

Feeding Frenzy

There were three times as many Trundles on April 1, 1934 as there were on February 1.

No one has ever figured out how Bugwumps know when the first day of the month is.

Homework Without Questions

Objective: The student will write equations of lines to model problems involving real-world and mathematical situations.

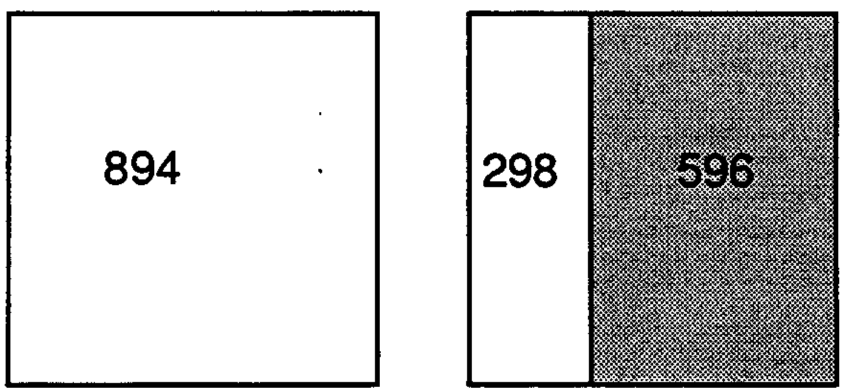
Materials: Problem situation and answers

Procedure: Read the problem situation. Manipulatives may be used to model the information provided. Select one of the answers and model the thinking and processing used.

Example: The school cafeteria serves hot lunches. You can purchase a complete lunch for $1.60 or just the main course for $.85. There are 894 students enrolled at the school and about two-thirds of them buy lunches each day.

What are the questions given the following answer (298).
Sample question: About how many students do not buy lunches each day? What calculations were done to arrive at that question? Possible responses: $894 - X = 298$ or $894 - 596 = 298$ or $894 - 2/3(894) = 298$ or $X - Y = 298$ or $X - 2/3X = 298$.

Provide time for students to explain their thinking using manipulatives or visuals whenever possible.



Questions generated will give insight into the thinking and processing students are experiencing. Discussion of the questions generated will provide opportunities for students to make clear distinctions, develop arguments, construct explanations and work with relatively complex understandings.

Developing activities like this allows students to see mathematics connected to their personal experiences. They will explore these connections in ways that create personal meaning. Higher order thinking skills such as making distinctions, applying ideas, forming generalizations, raising questions, and not just reporting experiences, facts, definitions, or procedures will be evident.

Research indicates sharing is best illustrated when participants explain themselves or ask questions in complete sentences and when they respond directly to comments of previous speakers. The dialogue builds coherently on participants' ideas to promote improved collective understanding of a theme or topic.

Homework Without Questions

The school cafeteria serves hot lunches. You can purchase a complete lunch for $1.60 or just the main course for $.85. There are 894 students enrolled at the school and about two-thirds of them buy lunches each day. Answers are given below. What are the questions?

Answers	Questions/Equations
1. 596	
2. $5.75	
3. 3 complete lunches and 2 main courses	
4. $953.60	
5. 298	
6. 2980 lunches	
7. 250 complete lunches	
8. 370 main courses	
9. Not more than 3	
10. Yes	

Doin' the Wave

After your class has finished "doin' the wave," enter the results in your graphing calculator. Use STAT, EDIT. Then look at the scatter plot for your data. Use STAT PLOT, GRAPH.

1. What is the relationship between the number of people and the amount of time it takes to complete the wave?

2. Choose two points from your data. Write the equation of a line through these two points.

3. Enter the equation you wrote for question 2 on the Y = Screen. GRAPH. How close is the line to your data points?

4. Now, use the calculator to write an equation for the line of best fit. Follow these steps:
   a. $\boxed{\text{STAT}}$ $\boxed{\text{CALC}}$ $\boxed{\text{LIN REG}}$
   b. Rounding off to the nearest hundredth, enter the equation in the y = screen
   c. $\boxed{\text{GRAPH}}$

5. Based on your graph, how long would it take 26 students to do the wave? ______

100 students? ______

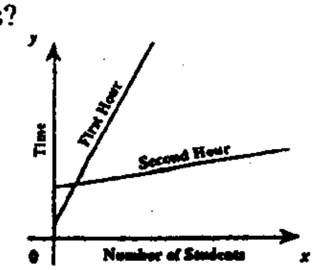
6. How many people could do the wave in 3 minutes?

7. Was your answer to question 6 a whole number? Does a non-whole number make sense for this answer?

8. How would your graph be different if every student stood up, raised their arms, and then turned around twice before sitting down?

9. Here are graphs from two classes who did the wave experiment. Why are their slopes different?

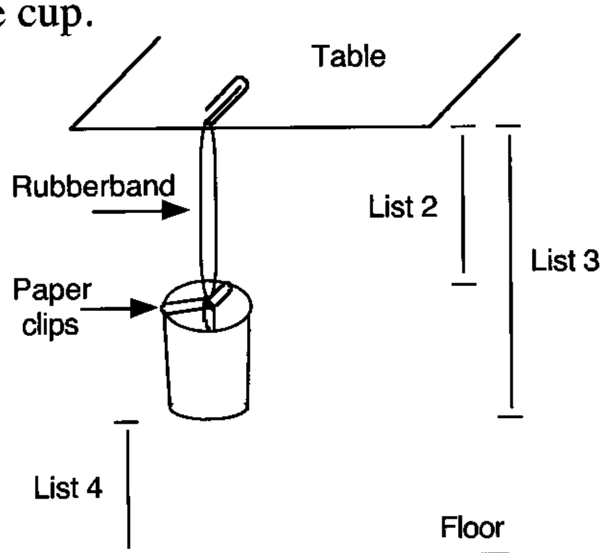
What could be a possible explanation for the difference in the y-intercepts?



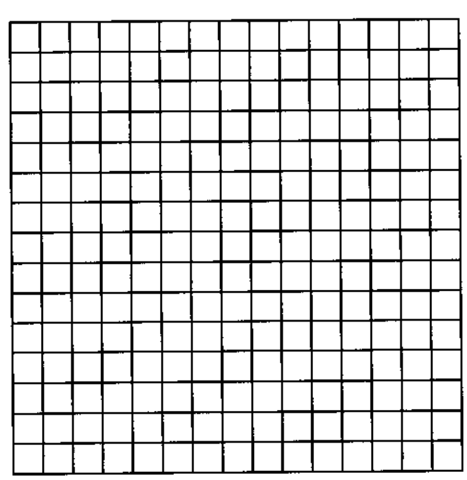
Activity 1: Stretch It

What is the relationship between the number of marbles in the cup and the distances shown below?

Use large paper clips and an 8 oz. paper cup to form a "hanging container." Measure the distances indicated below as marbles are added to the cup.



1. **Predict** a graph of the relationship between the number of marbles in the cup and the distances shown above. You should predict 3 different graphs.



2. Measure the indicated distances and record in the table below. Next, add 5 marbles to the container and measure each distance as before. Record the new measurements. Continue this process: add 5 marbles to the container, measure, and record.

Number of marbles List 1	Distance from table to top of cup (cm) List 2	Distance from table to bottom of cup (cm) List 3	Distance from bottom of cup to floor (cm) List 4
0			
5			
10			
15			
20			
25			
30			

Trend line #1: Consider the relationship between the distance from the table to the top of the cup (List 2) and the number of marbles (List 1.)

3. Create a scatter plot using a graphing calculator.
4. Estimate a rate of change by finding first differences in your data.

5. Estimate the y-intercept (*starting point.*)

6. Use the estimated rate and y-intercept to find a trend line for your data.

7. Graph the trend line over the scatter plot. Adjust the parameters *y-intercept* and *rate of change*, if necessary, for a better fit.

8. What are the units of slope for the trend line?

9. What is the meaning of the y-intercept in the trend line?

Trend line #2: Consider the relationship between the distance from the table to the bottom of the cup (List 3) and the number of marbles (List 1.)

10. Create a scatter plot using a graphing calculator.

11. Estimate a rate of change by finding first differences in the data.

12. Estimate the y-intercept (*starting point.*)

13. Use the estimated rate and y-intercept to find a trend line for your data.

14. Graph the trend line over the scatter plot. Adjust the parameters *y-intercept* and *rate of change*, if necessary, for a better fit.

15. What are the units of slope for the trend line?

16. What is the meaning of the y-intercept in the trend line?

Trend line #3: Consider the relationship between the distance from the floor to the bottom of the cup (List 4) and the number of marbles (List 1.)

17. Create a scatter plot using graphing calculator.
18. Estimate a rate of change by finding first differences in the data.

19. Estimate the y-intercept (*starting point.*)

20. Use the estimated rate and y-intercept to find a trend line for your data.

21. Graph the trend line over the scatter plot. Adjust the parameters *y-intercept* and *rate of change*, if necessary, for a better fit.

22. What are the units of slope for the trend line?

23. What is the meaning of the y-intercept in the trend line?

Activity 2: Comparing Graphs

1. Graph all three scatter plots and trend lines on your calculator in the same window. Sketch:

2. Write the equations of the trend lines.
   Trend line #1, List 2 vs List 1:
   Trend line #2, List 3 vs List 1:
   Trend line #3, List 4 vs List 1:

3. Compare the slopes of the trend lines. What do you find?

4. Find the difference between the y-intercept of trend line #1 and the y-intercept of trend line #2. Where is this distance in the experiment?

5. Use your trend line to determine when the cup will touch the floor. Describe your strategy.

6. Use your trend lines to determine how far the top of the cup would be from the table if you added 42 marbles. Describe your strategy.

Linear Regression with Coded Data

Updated: 01/18/08

Objective:
Students will make predictions using regression lines.

Connections to Previous Learning:
Students should be able to code data, create scatterplots, and find least square lines using a graphing calculator.

Connections to AP*:
AP Statistics Topic: Linear Bivariate Data

Time Frame:
30 minutes

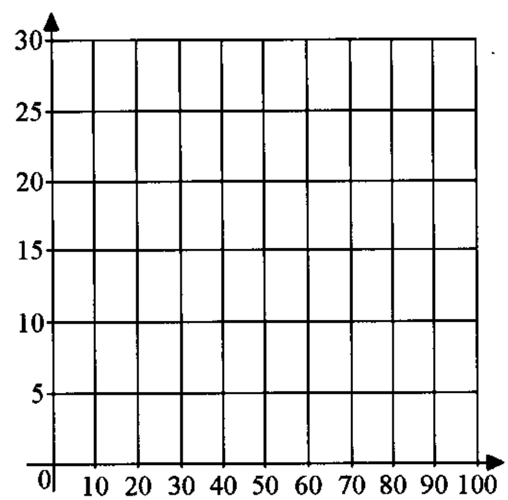
Materials:
Student activity pages, graphing calculator

Teacher Notes:
If this is the students' first experience with linear regression, they should be encouraged to manually fit a line to the data. The answers for the lesson are written using the least squares regression lines using a graphing calculator. The data in the lesson has been coded, so students must write the responses using the actual units

Linear Regression with Coded Data

1. The following table shows the birth rates (per 1000 people) in the United States since 1920. (The variable year is the number of years since 1920.)

Year	10	20	30	40	50	60	70
Birth rate	21.3	19.4	24.1	23.7	18.4	15.9	16.7



a) Construct a scatterplot for the data and label the axes.

b) What year is represented by year 40?

c) Determine a line that models the data and identify the variables.

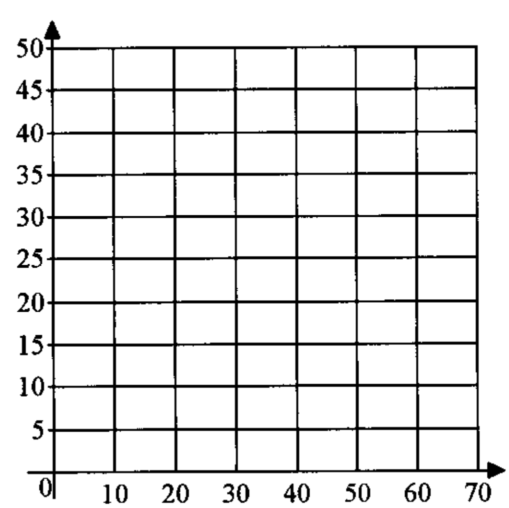
d) In the context of the problem, what does the y-intercept represent?

e) In the context of the problem, what does the value of the slope represent?

f) Using the line that models the data, predict the birth rate for the year 2000.

2. Suppose you work at the stadium of a major league baseball team. Your job is to order food for the concession stands. In order to determine the number of hotdogs that you need to order for the next game, you examine data from previous games. You have found that the expected attendance at a game can be determined from advance ticket sales. The data below show the number of advance tickets sold (in thousands) and the number of hotdogs purchased (in thousands) at each game.

Ticket Sales	45	64	37	58	41	29
Hotdogs sold	32	46	25	44	32	18



a) Construct a scatterplot for the data and label the axes.

b) Determine a line that models the data and identify the variables.

c) In the context of the problem, what does the value of the slope represent?

d) If advance ticket sales are 52,000, approximately how many hotdogs would you expect to sell at the game?

3. A real estate agent must often determine the price of a house that is to be sold. One of the factors that helps determine the cost of the house is its size, or the number of square feet of living space in the house. The data below gives the number of square feet (in hundreds) and the cost (in thousands of dollars) of houses that sold recently in a town.

Size of house	15	31	22	18	26	14	35	25	23
Cost of house	130	280	195	160	225	140	315	210	205

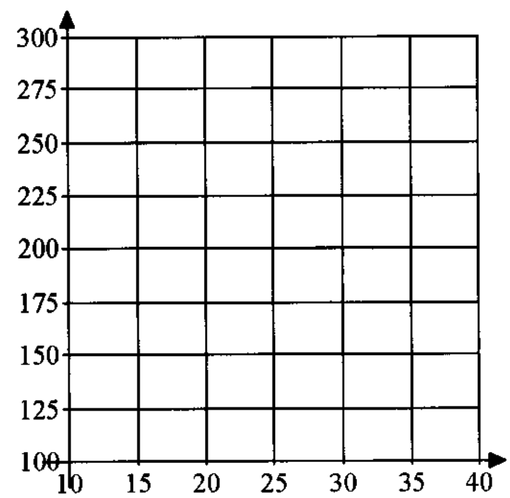
a) Construct a scatterplot for the data and label the axes.

b) Determine a line that models the data and identify the variables.

c) In the context of the problem, what is the meaning of the slope?

d) What is the approximate cost of a house that has 2000 square feet of living space? 3000 square feet?

e) If a house cost $260,000, approximately how many square feet would you expect it to have?



4. Prior to 2001, the Leaning Tower of Pisa was leaning more and more each year. The data that follows gives the lean of the tower for the years 1975 to 1987. The lean is the distance (in meters) between where a point on the tower would be if the tower were straight and where it really is. In 1975, the lean of the tower was 2.9642 meters and appears as 642 (the last three digits of 2.9642) in the table.

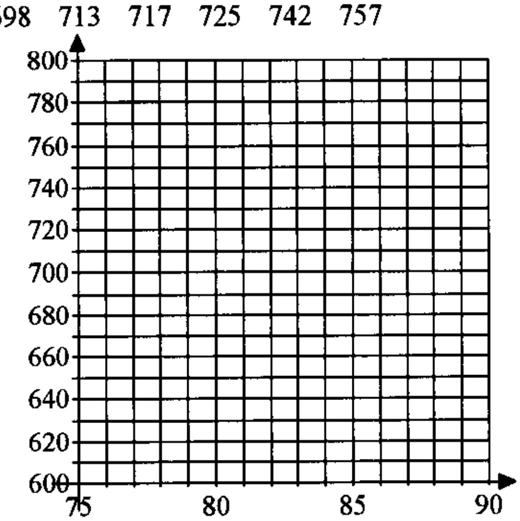
Year	75	76	77	78	79	80	81	82	83	84	85	86	87
Lean	642	644	656	667	673	688	696	698	713	717	725	742	757

a) Construct a scatterplot for the data and label the axes.

b) In 1987, the lean is coded as 757. What does the number 757 represent?

c) Determine a line that models the data and identify the variables.

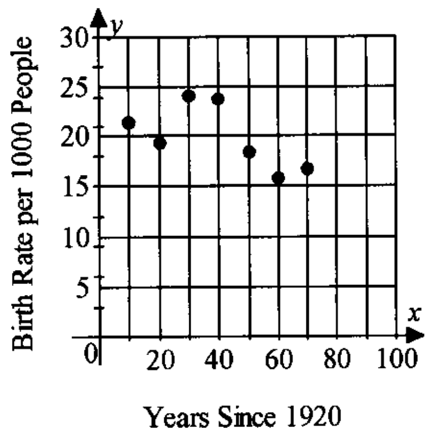
d) Predict the lean of the tower for the year 1990.



Linear Regression with Coded Data

Answers:

1. a)



b) $1920 + 40 = 1960$

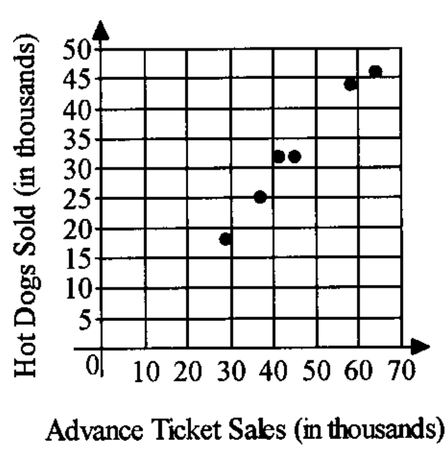
c) $y = -.095x + 23.714$ (least squares regression line). If students fit the line manually, their answers should be close to this line. The variable x represents the number of years since 1920, and the variable y represents the birth rate per 1000 people.

d) The birth rate in the year 1920 was approximately 23.714 babies per 1000 people.

e) For each increase of one year, there is approximately a .095 decrease in births per thousand people.

f) $x = 2000 - 1920 = 80$, If $x = 80$, then $y = 16.143$ births per1000 people.

2. a)

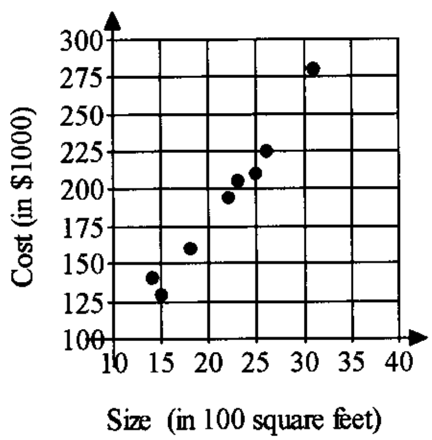


b) $y = .809x - 4.123$ (least squares regression line). If students fit the line manually, their answers should be close to this line. The variable x represents the advance ticket sales in thousands and the variable y represents the number of hot dogs sold in thousands.

c) For an increase of 1000 advance ticket sales, there is an approximate increase of 809 (0.809×1000) hotdogs purchased at the game.

d) When 52,000 advanced tickets are sold, expect to sell 37,959 hotdogs.

3. a)



b) $y = 8.700x + 4.625$ (least squares regression line) The variable x represents the size of the house in 100 square feet, and the variable y represents the cost of the house in $1,000.

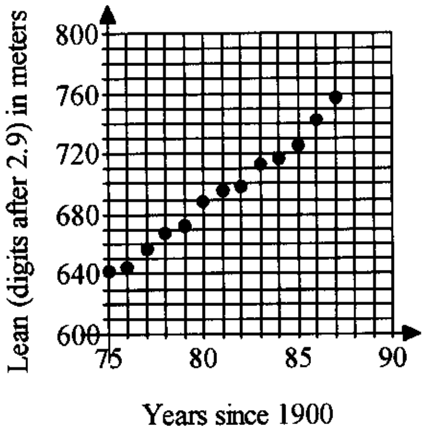
c) For an increase of 100 square feet in living space, there is an approximate increase of $8,700 in the cost of the house.

d) The value of y is 178.632, so the approximate cost is $178,632.

The value of y is 265.636, so the approximate cost is $265,636.

e) Since $x = 29.352225$ when $y = 260$, the living space is approximately 2935.225 square feet.

4.a)



b) In 1987, the tower was 2.9757 meters from where it would be if it were vertical.

$y = 9.318x - 61.121$ (least squares regression line)

The variable x represents the years since 1900, and the variable y represents amount of lean in $\frac{1}{10{,}000}$ of a meter more than 2.9 meters.

c) The predicted lean of the tower in 1990 is approximately 2.9777 meters.

GO FISH

How do wildlife biologists determine the number of fish in a lake or birds in a forest? Surely, they don't count them one by one. Follow the steps below to simulate a technique called "tag and recapture" that is used by naturalists to count large populations.

1. Each group has a lake (big baggie) full of fish (beans), and a net(a small cup). Use your "net" to take out a sample (about 1/2 cup) of fish.

2. Count the number of fish you have netted in your sample. "Tag" the netted fish using the permanent marker. Count and record the **"total number of tagged fish in lake"** and then return them to the lake. Be careful not to let any of the fish jump out on to the floor.

3. Gently shake the bag to thoroughly mix all the fish in the lake. Use your net to take another sample of fish. Count the number of tagged fish in the sample and record this number as **"number of tagged fish in sample."** Count and record the **"total number of fish in sample,"** also.

4. You now have three pieces of information: **the total number of tagged fish in the lake, the number of tagged fish in the sample, and the total number of fish in the sample**. Use this information and your knowledge of ratios to write and solve a proportion that will estimate the number of fish in the lake.

5. Return your sample of fish to the lake, gently mix the fish, and take another sample. Repeat steps 4 and 5 to get another estimate of the lake's fish population. Note the **number of tagged fish in the lake** is the same for every trial.

6. As a marine biologist it is important to get as accurate a count as possible, but each time you net a sample it costs the taxpayers $500 for your time and equipment. So far your samples have cost $1000. You will probably want to do at least one more sample, maybe more, before you will feel confident that your estimate is accurate.

7. When you are satisfied that you have done enough samples, record your results on the class data chart. Since your group can only submit one estimate, you will need to decide how to "average" your data. There are two ways. One way is to find the average of your estimates and use that as your final estimate. The other way is to treat the samples as one large sample and calclulate your estimate from that.

8. After your data is recorded on the class chart, count the fish in your lake to find the actual population and calculate the percent of error for your estimate. Record these two numbers on the class data chart.

Group Data Sheet

Sample	1	2	3	4
Total # of tagged fish in lake				
# of tagged fish in sample				
Total # of fish in sample				
Estimated # of fish in lake				

Use the space below to show your equations and computations.

1. What was your group's percent of error? _____

2. Use the information from the class data sheet to evaluate the work of other groups. Does there appear to be a correlation between the number of trials and the percent of error?

   ____

3. What are possible steps you could take to reduce the percent of error? __________

   ____

4. What do you think of this method for estimating fish populations? ______________

   ____

5. In order to determine the number of fish in Bachman Lake, a marine biologist collects 100 fish, tags them, and returns them to the lake. Then she collects another sample containing 25 fish and notes that 7 are tagged. If she solves a proportion based on this information, her estimate of the number of fish in the lake would be ______. How close to the actual number of fish in the lake do you think this estimate is?

   ____

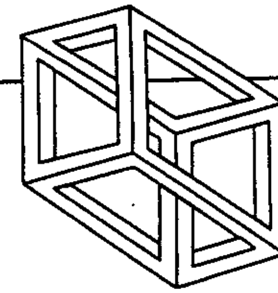
6. Game wardens need to keep a close watch on deer populations in order to put limits on hunters that will insure the population remains stable. Suggest an appropriate method for detrmining the deer population of a particular hunting area?

   ____
   ____
   ____

CLASS DATA SHEET

GROUP NAME	ESTIMATE	# SAMPLES	ACTUAL COUNT	% ERROR

Restless Rectangles



Institute Notes

Concept: Investigate the constants of proportionality within similar shapes as "shape ratios" and the scale factors between pairs of similar shapes as "size ratios" for a set of similar rectangles and use these ratios to solve problems.

TEKS Focus: **6.3**—The student solves problems involving proportional relationships.
7.3—The student solves problems involving proportional relationships.
8.3—The student identifies proportional relationships in problem situations.

Also:
Grade 6
4A, 11D, 12A, 13A
Grade 7
7A, 13C, 13D, 14A, 15A
Grade 8
7D, 14C,D, 15A, 16A, 16B

Overview: Participants will sort a group of rectangles cut from cardstock into two groups: those that have the same shape and those that do not have the same shape as the first group. The rectangles from each group are measured and their dimensions are recorded in Tables 1 and 2. The data is graphed and observations are made to determine which graphs represent a proportional relationship. Comparisons are made between any two rectangles from Group 1 (same shape group), and participants make generalizations about different ways to describe the shape and relative sizes of these rectangles. Through these investigations, participants identify the constants of proportionality within a set of similar shapes as "shape ratios" and the scale factors between pairs of similar shapes as "size ratios" and use these ideas to determine whether two geometric figures are similar or not.

Materials:
Restless Rectangles on cardstock
Scissors
1" graph paper
Cm Grid Paper
Metric rulers
Markers and/or peel-and-stick dots
Tape
Construction paper in two colors

Restless Rectangles

Procedure:

1. Participants are to work in groups of 2 to 4 following the procedure outlined in steps a - h in Activity 1.
2. The data collected from Activity 1 sheet is to be recorded in the tables provided in Activity 2. Have participants discuss patterns they see in the data. (See *Math Notes* for Activity 2).
3. Have participants graph the data for the rectangles in Group 1 on the grid in Activity 3 or on 1" graph paper and answer the questions at the end of the activity.
   Participants should be encouraged to study Table 1 and Graph 1 to investigate the relationship between the data in the table and the graph of the ordered pairs (W,L). This connection should reinforce the definition of a ratio as "*an ordered pair of measurements.*"
4. Have participants graph the data for the rectangles in Group 2 on the grid in Activity 4 or on 1" graph paper and answer the questions at the end of the activity.

Debriefing: Ask participants to use the data from their tables and graphs to make generalizations about the rectangles in Group 1.

Extensions: a. Provide participants with a set of triangles or other

Math Notes:
Participants should observe that these rectangles have the same shape because of the constant ratio $\frac{L}{W}$, or "shape ratio," obtained from Table 1, and that the corresponding sides are proportional. They should also conclude that the scale factor or "size ratio" between each pair of rectangles describes the relative sizes of any two rectangles in the set. It would be appropriate at this time to label the rectangles in Group 1 as *similar figures* and discuss the attributes of similar figures.

Restless Rectangles

geometric figures that have the same shape and some that do not. Have them sort the geometric figures and repeat this activity.

b. Ask participants to make a conjecture about the ratio of the lengths of the diagonals for any two similar rectangles from Group 1. Then, have them measure the diagonals of two rectangles from Group 1 and compare their ratio to the ratio formed by lengths of corresponding sides.

c. Suppose the rectangle with sides 4 cm and 6 cm is cut in half to form two smaller rectangles. Would the smaller rectangle be similar to the original? Why or why not?

Assessment: 1. Have participants make a set of geometric figures that have the same shape and explain how they determined this.

2. Have participants apply their knowledge of scale factor between similar figures to answer the questions in Activity 5.

Notes:

b. Participants should recognize that the ratio of the diagonals for any two similar rectangles is the same as the ratio of their corresponding sides. Example: The diagonal of the rectangle with dimensions 4 cm by 6 cm has a length of $\sqrt{52}$ cm and the rectangle with dimensions 8 cm by 12 cm has a diagonal of length $\sqrt{208}$, obtained by using the Pythagorean Theorem. The ratio of the diagonals can be expressed as $\sqrt{52}:\sqrt{208}$. When this ratio is simplified to $2\sqrt{13}:4\sqrt{13}$ or 1:2, a comparison can more readily be observed. This is the same as the ratio of two corresponding sides of the two rectangles expressed as 4:8 or 6:12. It is important to introduce now this concept of the constant ratio of corresponding parts of similar figures, since the next activity, *One Size Fits All*, will explore this idea in depth and extend to the ratio of the perimeters of similar figures (equal to the ratio of their corresponding parts) and the ratio of areas of similar figures (equal to the square of the ratio of their corresponding parts).

Restless Rectangles

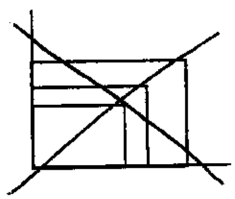
Restless Rectangles

Activity 1

a. Cut out each rectangle.

b. Sort the rectangles into 2 groups:
Group 1: Rectangles that have the same shape.
Group 2: Rectangles that do not have the same shape as those in Group 1.

c. Arrange the rectangles in Group 1 in order from largest to smallest and place the largest on the Restless Rectangles Guide first followed by the next largest on top, etc. as shown below.



d. Select the largest rectangle and place the edge of a ruler along its diagonal. What do you observe?

e. Are there any rectangles that you put in Group 1 that you now think belong to Group 2? If so, put them in Group 2.

f. Repeat steps C and D above for the rectangles in Group 2. What do you observe about their diagonals?

g. How could you use the information about the diagonals of the rectangles in Group 1 to show that they have the "same shape?"

h. Write a statement about your observations concerning rectangles that have the same shape.

Answers and Math Notes:

a. Dimensions of rectangles:
A: 3 cm x 2 cm; G: 6 cm x 4 cm; I: 6 cm x 9 cm; D: 8 cm x 12 cm; E: 12 cm x 6 cm; F: 5 cm x 3 cm; C: 3cm x 4 cm; B: 10 cm x 4 cm; H: 7 cm x 5 cm

b. Participants will sort the rectangles according to "same shape" without measuring. Ask participants to explain the criteria used to sort the rectangles into two distinct groups.

c. Ask participants to observe the arrangement of the rectangles from Group 1 on the Guide. Ask them if they would like to change any of the rectangles placed in Group 1 or Group 2 and why.

d. Participants should observe that the diagonals of all rectangles in Group 1 lie on the same line through the origin. If there is any rectangle whose diagonal is not aligned with the others, its placement into this set should be reviewed.

e. These diagonals do not lie on the same straight line through the origin.

g. Arrange these rectangles on centimeter grid paper with the width and length aligned along the x- and y-axes, respectively. Observe that the diagonals lie along a line through the origin. This means that the widths and lengths of the rectangles in this set form a proportional relationship.

h. Rectangles that have the same shape will have their diagonals all on the same line that passes through the origin.

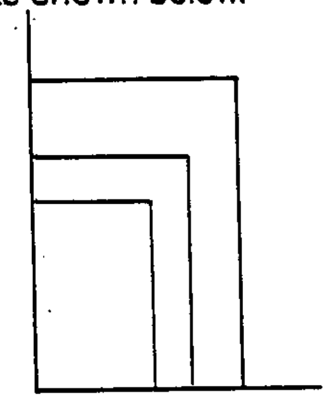
Restless Rectangles

Activity 1

a. Cut out each rectangle.

b. Sort the rectangles into 2 groups:
Group 1: Rectangles that have the same shape.
Group 2: Rectangles that do not have the same shape as those in Group 1.

c. Arrange the rectangles in Group 1 in order from largest to smallest and place the largest on the Restless Rectangles Guide first followed by the next largest on top, etc. as shown below.



d. Select the largest rectangle and place the edge of a ruler along its diagonal. What do you observe?

e. Are there any rectangles that you put in Group 1 that you now think belong to Group 2? If so, put them in Group 2.

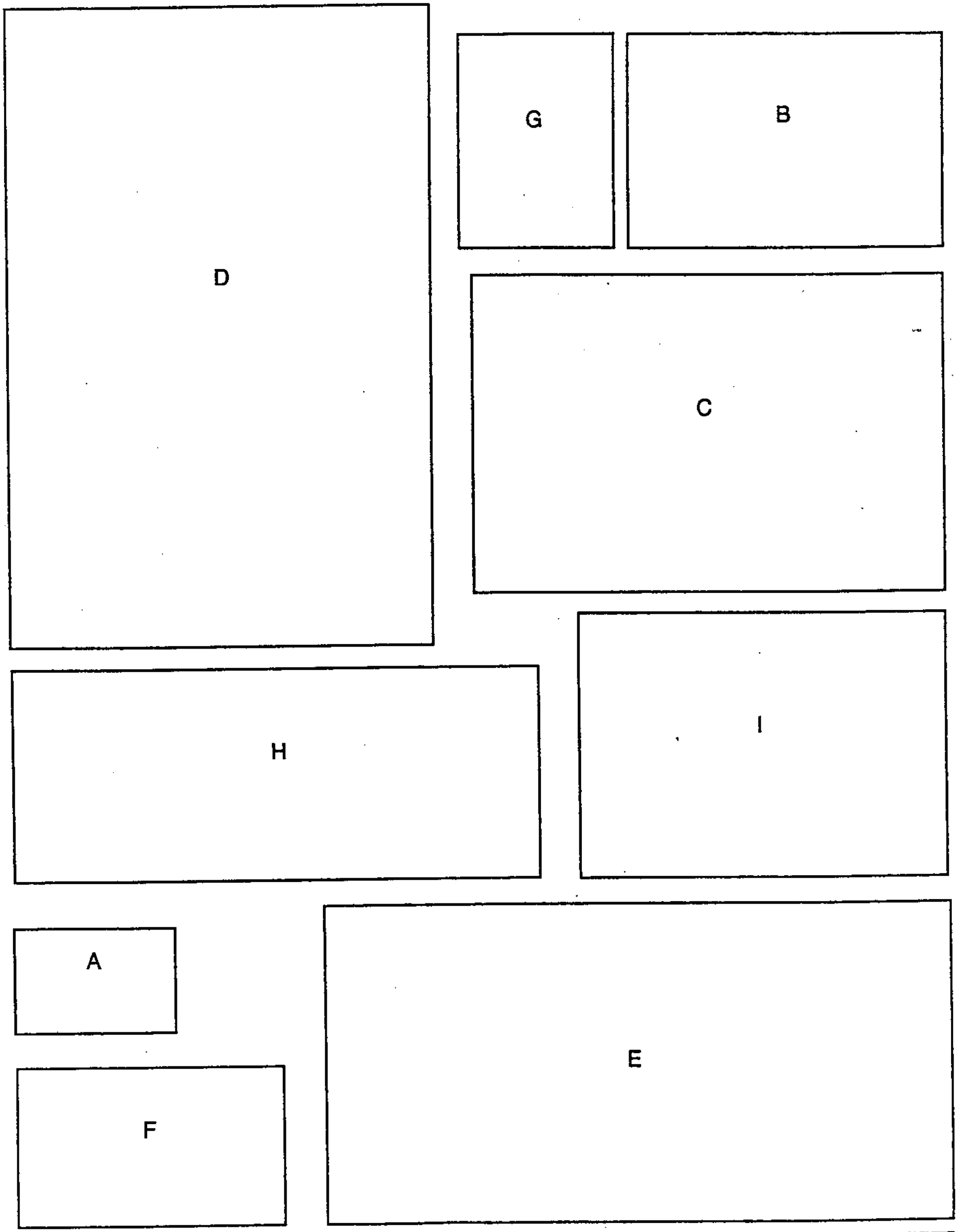
f. Repeat steps C and D above for the rectangles in Group 2. What do you observe about their diagonals?

g. How could you use the information about the diagonals of the rectangles in Group 1 to show that they have the "same shape?"

h. Write a statement about your observations concerning rectangles that have the same shape.

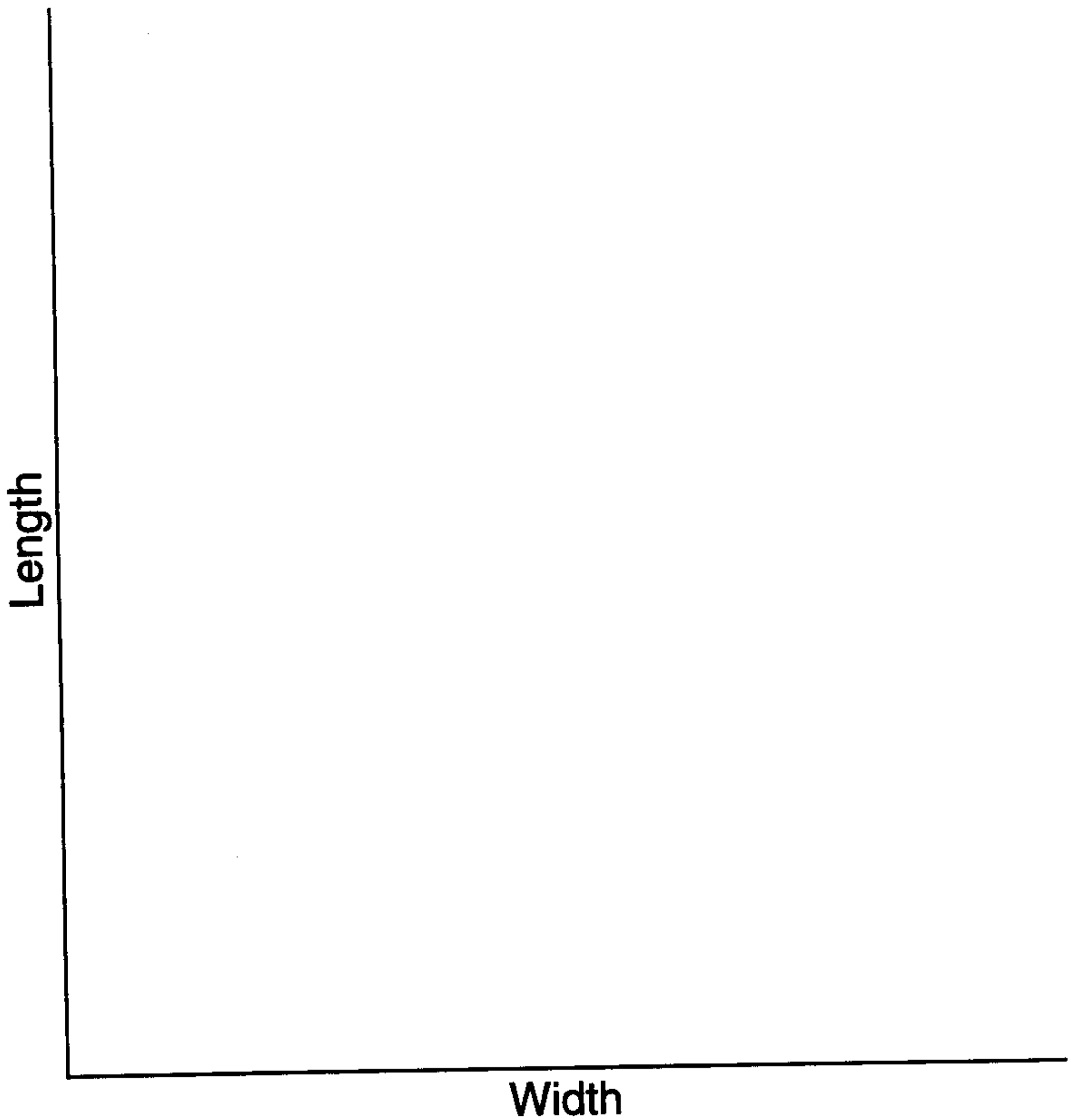
Restless Rectangles

Activity 1



Restless Rectangles

Guide for Activity 1



Restless Rectangles

Restless Rectangles

Activity 2

Use a centimeter ruler to measure the length and width of each rectangle in Group 1 and record in the table below. Do the same for the rectangles in Group 2.

Table 1—Group 1 Data

Rectangle	W	L
A	2	3
G	4	6
I	6	9
D	8	12

Table 2—Group 2 Data

Rectangle	W	L
E	6	12
F	3	5
C	3	4
B	4	10
H	5	7

a. What is the ratio of L:W for the rectangles in Group 1?

b. Is this a constant ratio?

c. How does this ratio describe the "shape" of these rectangles?

d. Compare the ratios of L:W in Group 2. Is there a constant ratio? Explain.

TEXTEAMS Rethinking Middle School Mathematics: Proportionality Activity-2

Reason and Communicate:
Ask participants to explain what it means to have a constant ratio in comparing L to W for the two groups.

Have participants write an equation that expresses the relationship that exists among the rectangles in Group 1. They should write one of the equations *L=kW* or *L/W=k*. Ask them to explain what *k* means in each equation.

Ask participants to compare the length and width of the smallest rectangle in Group 1 to the largest. Have someone explain how to describe the relative sizes of these two rectangles. They should note that the dimensions of the largest rectangle are *c* times as large as the dimensions of the smallest rectangle and that *c* is a scale factor that describes the relative sizes of the two rectangles.

Answers:
a. L:W = 3:2
b. Yes
c. The ratio of the length to the width determines the shape of the rectangle.
d. No, the ratios vary. The rectangles do not have the same shape.

Math Notes:
Ask participants to compare the data in Tables 1 and 2 and note any likenesses or differences. They should recognize that the ratio of L:W in Table 1 is a constant ratio, while the ratio L:W in Table 2 is not constant. The constant ratio in Table 1 can be represented by *k*, called the constant of proportionality. This constant describes the "shape" of every rectangle in Group 1. The equation $L/W = k$ can be expressed as $L = kW$, where L is a constant multiple of W. The constant of proportionality, *k*, can be thought of as the "shape ratio" for this set of similar rectangles (*Dick Stanley, Dana Center of The University of California at Berkley*).

By comparing the lengths and widths of the smallest and largest rectangles from Group 1, participants should recognize that the dimensions of the largest rectangle are 4 times those of the smallest rectangle. We can write an equation that relates the length L_2 of the largest rectangle to L_1, the length of the smallest rectangle as $L_2 = 4L_1$. Ask participants to write an equation relating the widths of the largest and smallest rectangle in Group 1 ($W_2 = 4W_1$). Have them do the same for any two rectangles from Group 1. Thus, for any two rectangles in Group 1, $L_2 = cL_1$ and $W_2 = cW_1$, where c is the *scale factor*. This constant *c* reflects the relative size of the dimensions of one rectangle when compared to the dimensions of another rectangle in our set of rectangles that have the same shape. (For example, $L_2 = 4L_1$ and $W_2 = 4W_1$ tell us that the dimensions of the largest rectangle are four times the dimensions of the smallest rectangle). The constant **c** can be thought of as the "size ratio" between these two rectangles. (*Dick Stanley, Dana Center of The University of California at Berkley*).

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Restless Rectangles

Activity 2

Use a centimeter ruler to measure the length and width of each rectangle in Group 1 and record in the table below. Do the same for the rectangles in Group 2.

Table 1—Group 1 Data

Rectangle	W	L

Table 2—Group 2 Data

Rectangle	W	L

a. What is the ratio of L:W for the rectangles in Group 1?

b. Is this a constant ratio?

c. How does this ratio describe the "shape" of these rectangles?

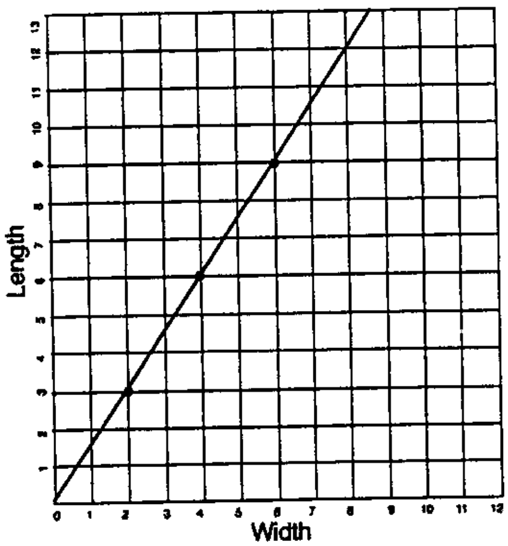
d. Compare the ratios of L:W in Group 2. Is there a constant ratio? Explain.

Restless Rectangles

Restless Rectangles

Activity 3—Group 1 Graph

Use the data from Table 1 to make a graph.



a. Use your ruler to draw a line through the points. What do you observe? Does the line pass through all the points? Does the line pass through the origin?

b. Would a rectangle with a length of 10 cm and a width of 8 cm belong in this group? Why or why not?

c. State the dimensions of another rectangle that would belong to Group 1.

TEXTEAMS Rethinking Middle School Mathematics: Proportionality Activity-11

Answers:
a. All the points should lie on a straight line passing through the origin.
b. No, because the coordinate (8,10) would not lie on this line.
c. Answers will vary. Example: 18 cm by 27 cm

Math Notes:
The graph of the data for the rectangles in Group 1 should lie on a straight line that passes through the origin. This result is consistent with a characteristic of a proportional relationship. (In other words, in this group of rectangles, the width is proportional to the length.) Ask participants to use this graph to give the dimensions of other rectangles that would belong to Group 1 and justify their answers. Have participants do a scatter plot of the data on a graphing calculator and a line of best fit. They could trace along the line or use a table to find dimensions of other rectangles that belong to Group 1, the group of similar rectangles that have the "same shape."

Reason and Communicate:
Ask participants to reflect on a previous activity where "groups of" objects were compared resulting in equivalent ratios (*Perfect Paint Color*). This activity builds upon the *Perfect Paint Color* activity by using "strips of three" and "strips of two" to establish a set of ordered pairs (W, L) whose ratios $\frac{L}{W}$ are equivalent.

Have participants reflect on the use of tables and graphs to represent the same data. The table helps one to make comparisons and find a constant ratio. This enables one to determine if there is a proportional relationship from a tabular form. By observing the graph of this relationship, participants should note that the vertices (W, L) from the table lie on a straight line that passes through the origin. Ask participants to state any observations about the equivalent ratios from their table and the ordered pairs graphed and write a definition of a ratio. They should observe that a ratio is an ordered pair of measurements.

While using the graph to interpret data, participants should be encouraged to consider questions like the following:
"Does a rectangle with dimensions W = 18 cm and L = 26 cm belong to Group 1? Why or why not?"
No, because the ratio of $\frac{L}{W}$ is not the same as the constant ratio $\frac{3}{2}$ for Group 1.

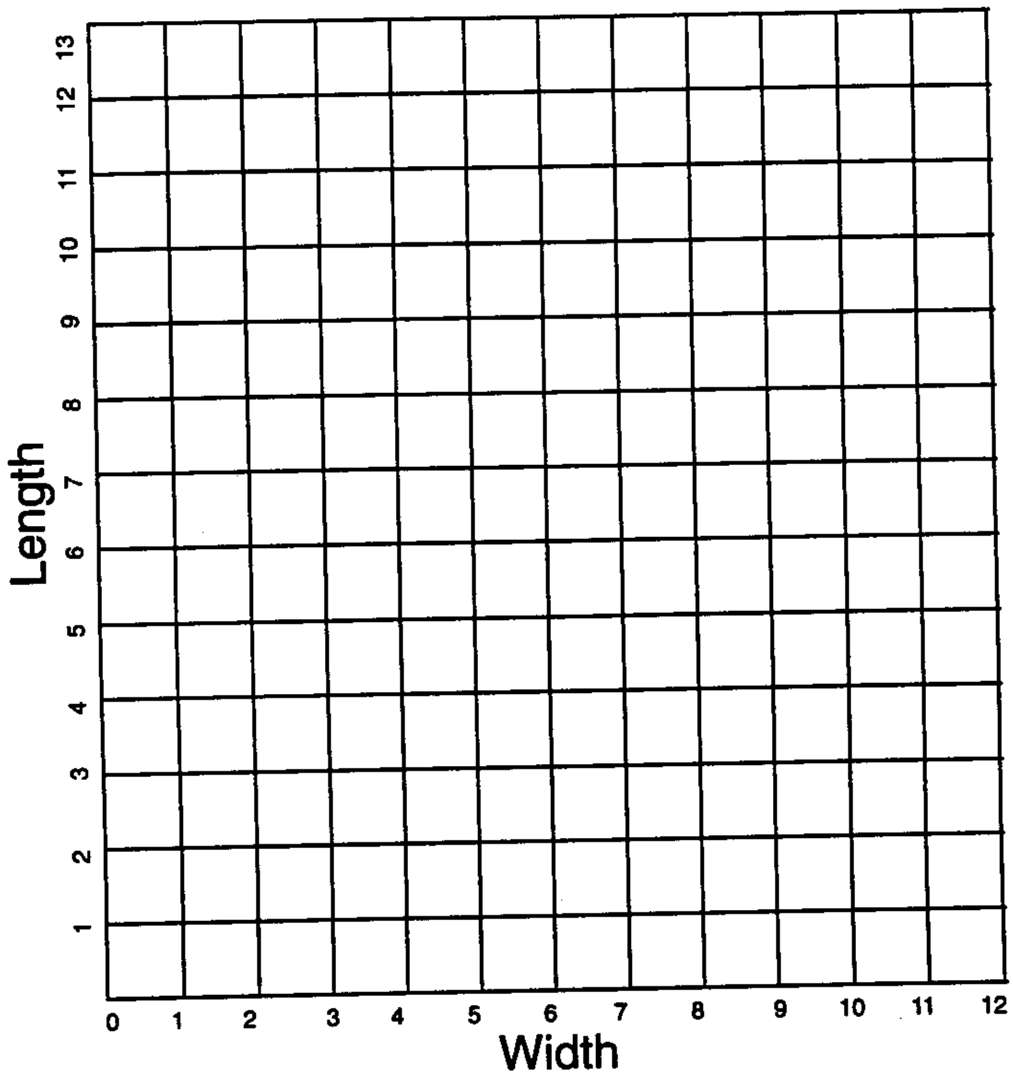
"What does the ordered pair (10,15) mean in the context of this problem?"
The width of the rectangle is 10 units and the length is 15 units.

"Given the ordered pair (W, 36), find the value of W so that the point will lie on the graph of the Group 1 data."
W=24

Restless Rectangles

Activity 3—Group 1 Graph

Use the data from Table 1 to make a graph.



a. Use your ruler to draw a line through the points. What do you observe? Does the line pass through all the points? Does the line pass through the origin?

b. Would a rectangle with a length of 10 cm and a width of 8 cm belong in this group? Why or why not?

c. State the dimensions of another rectangle that would belong to Group 1.

Restless Rectangles

Restless Rectangles

Activity 3—Group 1 Graph continued

d. Place each rectangle from Group 1 on the grid so that the length and width are aligned with the axes and trace. State your observations.
Cut out one 3 cm by 15 cm strip from one color of construction paper. Cut out another strip 2 cm by 15 cm from a different color. Use the wider strip to mark off 3 cm sections along the vertical axis. Use the other strip to mark off 2 cm sections along the horizontal axis.

e. How many groups of 3 cm are there in the length of rectangle A? rectangle G? How many groups of 2 cm are there in the width of rectangle A? rectangle G?

f. Use these strips to compare groups of 3 cm and 2 cm for the other rectangles in Group 1. Do these rectangles seem to be "growing" in the same way as you move from smallest to largest? Explain.

g. The dimensions of rectangle G are how many times the dimensions of rectangle A? This factor is called a scale factor.

h. Write an equation that describes the relationship between the length, L_2, of rectangle G and the length, L_1, of rectangle A. Write another equation that describes the relationship between the width, W_2, of rectangle G and the width, W_1, of rectangle A.

i. What seems to be the same in each equation? What is another name for this constant?

j. What does this constant describe about rectangle A and rectangle G? What would the ratio between their lengths describe about any two rectangles in Group 1?

k. Select any two rectangles from Group 1 (other than A and G) and identify the scale factor from one to the other for a "sizing up" and "sizing down" situation.

TEXTEAMS Rethinking Middle School Mathematics: Proportionality Activity-141

Answers:
d. The endpoints of a diagonal of each rectangle lie on the line.
e. Rectangle A: 1 group of 3 cm, 1 group of 2 cm
Rectangle G: 2 groups of 3 cm, 2 groups of 2 cm
f. Yes, the lengths and widths of any two rectangles in Group 1 "grow" by the same scale factor.
g. 2 times
h. $L_2=2L_1$, $W_2=2W_1$

i. In each equation, there is a constant factor 2 called the scale factor.
j. This constant "c" for any two rectangles in Group 1 is also referred to as the "size ratio".
k. Example: What is the scale factor from the rectangle with *w:l of 4:6* to the rectangle with *w:l of 10:15*?
$\frac{5}{2}$
What scale factor would represent a reduction of the rectangle with sides 10 cm and 15 cm to a rectangle with sides 2 cm and 3 cm?
$\frac{1}{5}$

Restless Rectangles

Activity 3—Group 1 Graph continued

d. Place each rectangle from Group 1 on the grid so that the length and width are aligned with the axes and trace. State your observations.
Cut out one 3 cm by 15 cm strip from one color of construction paper. Cut out another strip 2 cm by 15 cm from a different color. Use the wider strip to mark off 3 cm sections along the vertical axis. Use the other strip to mark off 2 cm sections along the horizontal axis.

e. How many groups of 3 cm are there in the length of rectangle A? rectangle G? How many groups of 2 cm are there in the width of rectangle A? rectangle G?

f. Use these strips to compare groups of 3 cm and 2 cm for the other rectangles in Group 1. Do these rectangles seem to be "growing" in the same way as you move from smallest to largest? Explain.

g. The dimensions of rectangle G are how many times the dimensions of rectangle A? This factor is called a scale factor.

h. Write an equation that describes the relationship between the length, L_2, of rectangle G and the length, L_1, of rectangle A. Write another equation that describes the relationship between the width, W_2, of rectangle G and the width, W_1, of rectangle A.

i. What seems to be the same in each equation? What is another name for this constant?

j. What does this constant describe about rectangle A and rectangle G? What would the ratio between their lengths describe about any two rectangles in Group 1?

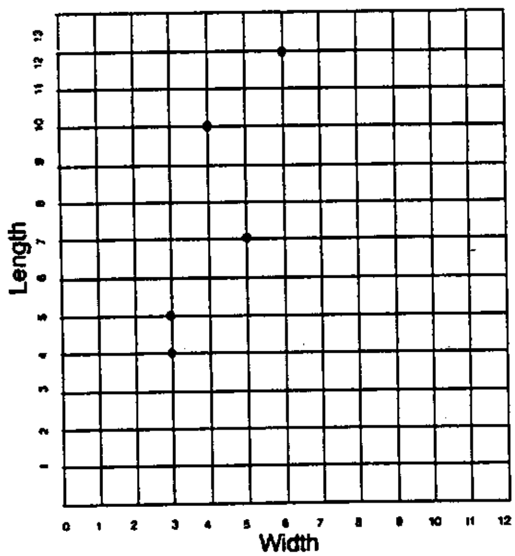
k. Select any two rectangles from Group 1 (other than A and G) and identify the scale factor from one to the other for a "sizing up" and "sizing down" situation.

Restless Rectangles

Restless Rectangles

Activity 4—Group 2 Graph

Use the data from Table 2 to make a graph.



a. Is it possible to draw a line through these points like you did in Graph 1?

b. Place each rectangle from Group 2 on the grid and trace.

c. What do you observe about these rectangles? Are they "growing" in the same way? Explain.

TEXTEAMS Rethinking Middle School Mathematics: Proportionality Activity-15

Reason and Communicate:
Participants should observe that the rectangles in this group have diagonals that do not line up on the same line. Ask them what this means. Ask participants to explain how they could use Table 2 to help them determine if this is a proportional relationship.

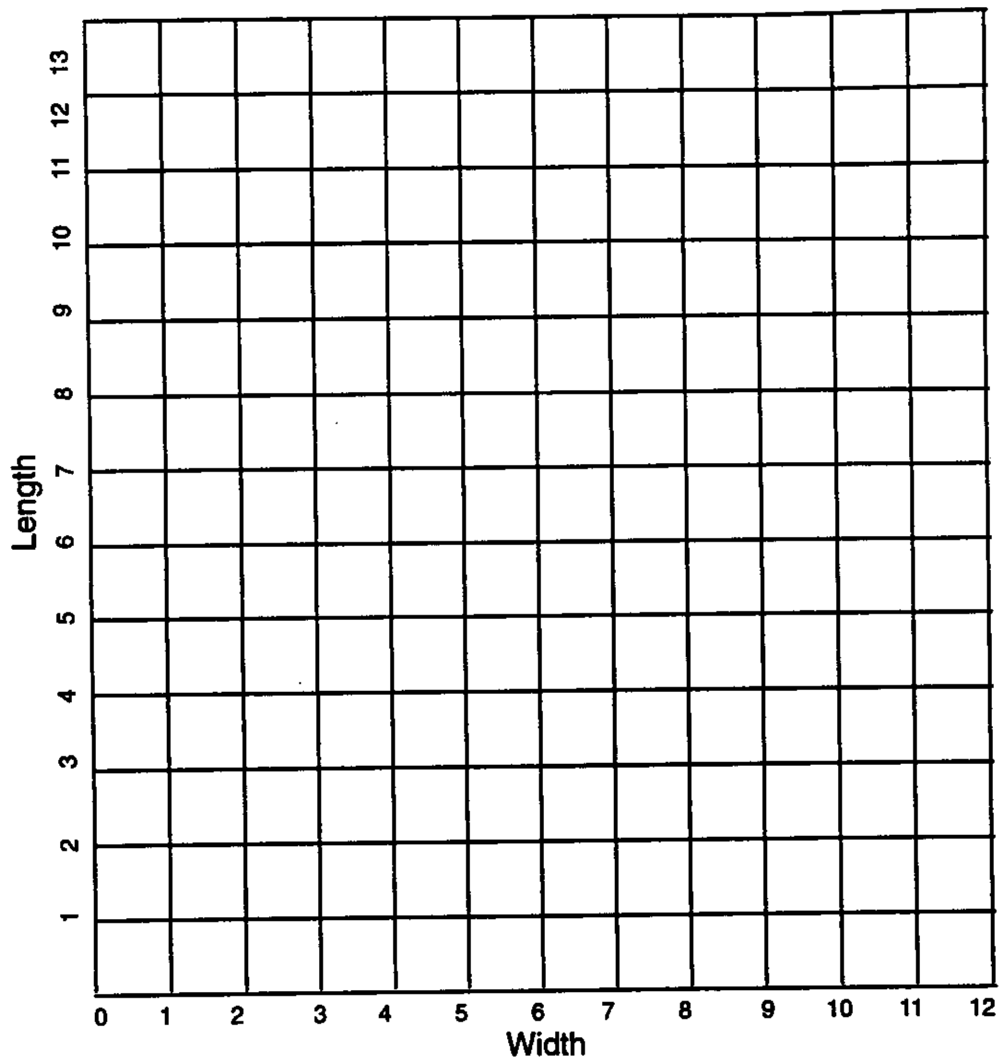
Answers:
a. No
c. The dimensions of the rectangle are not growing in the same way. The diagonals do not lie on the same line.

Math Notes:
The graph of the data for Group 2 rectangles does not represent a straight line. One can conclude that the data for Group 2 does not represent a proportional relationship. This conclusion can also be verified by examining the data in Table 2. There is not a constant ratio when comparing L to W.

Restless Rectangles

Activity 4—Group 2 Graph

Use the data from Table 2 to make a graph.



a. Is it possible to draw a line through these points like you did in Graph 1?

b. Place each rectangle from Group 2 on the grid and trace.

c. What do you observe about these rectangles? Are they "growing" in the same way? Explain.

Restless Rectangles

Restless Rectangles

Activity 5

Use your Restless Rectangles to solve these problems involving a scale factor.

a. Identify the scale factor from rectangle G to rectangle C.

b. What is the scale factor from rectangle I to rectangle A? Did this scale factor create an enlargement or reduction from I to A?

c. Is there a scale factor from rectangle D to rectangle E? Explain.

d. Use centimeter grid paper to trace rectangle D. Then draw rectangle J so that the scale factor from rectangle D to rectangle J is 3. State the dimensions of rectangle J. How do these dimensions compare to those of rectangle D?

e. Trace rectangle C on centimeter grid paper. Next, draw Rectangle L so that the scale factor from rectangle C to rectangle L is 1/3. What are the dimensions of rectangle L? How do these compare to those of rectangle C?

f. If you use a scale factor (less than one) from figure R to figure S, then the result is a(n)________________. (enlargement , reduction)

g. If you use a scale factor (greater than one) from figure W to figure Z, then the result is a(n) ________________. (enlargement, reduction)

h. If you use a scale factor of 1 from figure P to figure Q, then the result is a ________________.

TEXTEAMS Rethinking Middle School Mathematics: Proportionality Activity-145

Answers:

a. There isn't one because the rectangles are not the same shape (i.e., they are not similar).

b. $\frac{2}{6} = \frac{1}{3}$, a reduction

c. No, because the two rectangles are not similar.

d. The dimensions of rectangle J are 24 units x 36 units; the dimensions of rectangle J are 3 times the corresponding dimensions of rectangle D

e. The dimensions of rectangle L are 1 unit x $\frac{4}{3}$ units; the dimensions of rectangle L are 1/3 the length of the corresponding dimensions of rectangle C.

f. reduction

g. enlargement

h. congruent figure

Restless Rectangles

Activity 5

Use your Restless Rectangles to solve these problems involving a scale factor.

a. Identify the scale factor from rectangle G to rectangle C.

b. What is the scale factor from rectangle I to rectangle A? Did this scale factor create an enlargement or reduction from I to A?

c. Is there a scale factor from rectangle D to rectangle E? Explain.

d. Use centimeter grid paper to trace rectangle D. Then draw rectangle J so that the scale factor from rectangle D to rectangle J is 3. State the dimensions of rectangle J. How do these dimensions compare to those of rectangle D?

e. Trace rectangle C on centimeter grid paper. Next, draw Rectangle L so that the scale factor from rectangle C to rectangle L is 1/3. What are the dimensions of rectangle L? How do these compare to those of rectangle C?

f. If you use a scale factor (less than one) from figure R to figure S, then the result is a(n)____________________. (enlargement , reduction)

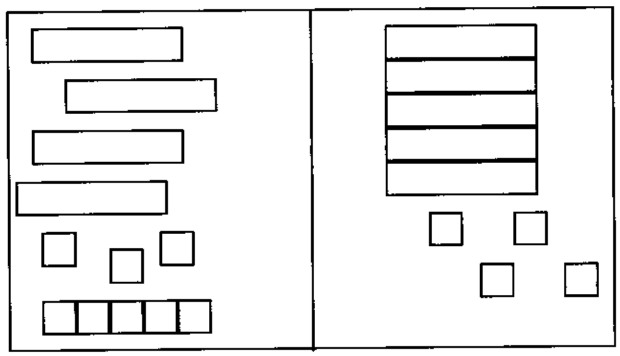
g. If you use a scale factor (greater than one) from figure W to figure Z, then the result is a(n) __________________. (enlargement, reduction)

h. If you use a scale factor of 1 from figure P to figure Q, then the result is a ______________________.

Activity 1: Concrete Models

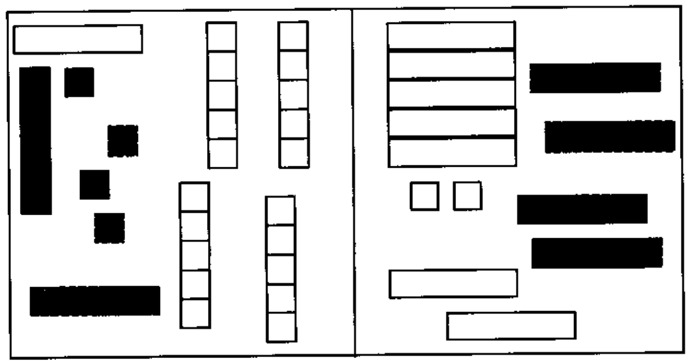
Use algebra tiles to solve each equation. Sketch each step and record the symbolic representation for each step.

1.



original equation

2.



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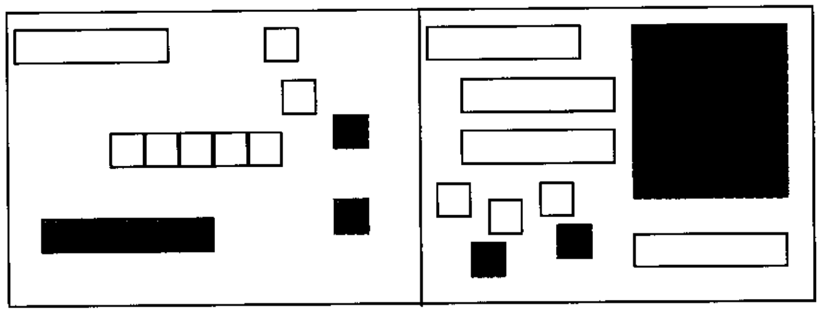
original equation

3.



____

original equation

Activity 2: Using Concrete Models

Build each equation and solve with algebra tiles. Record the intermediate steps and the solution.

1. $8x - 12 = 3x + 13$

2. $-6 + x = 2x + 4 - 3x - 10$

3. $3x + 7 - (3 - x) = (x + 2) + x$

4. $2x - 9 - (3x + 1) = 5x - (4x - 2)$

Activity: Stays the Same

Solve each of the following problems, showing each step, in the three ways below. Sketch the algebra tile solution and the graphical solution.

1. $7 + 3x = -8 - 2x$

Algebra Tile Solution:	Algebraic Solution:	Graphic Solution:

2. $x + 5 = 1 - x$

Algebra Tile Solution:

Algebraic Solution:

Graphic Solution:

3. $3-2x=x-6$

Algebra Tile Solution:

Algebraic Solution:

Graphic Solution:

4. $-2x + 4 = -5 + 4x$

Algebra Tile Solution:	Algebraic Solution:	Graphic Solution:

TABLE 1
Example 1: "I Know Your Favorite Number!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel's Favorite Number is 7)	Private Eye's Work (Lyta Sets Joel's Number as x)
1. Think of your favorite number.	7	x
2. Multiply it by 3.	×3 21	$3x$
3. Add your favorite number and 1 to it.	29	$3x+x+1$ $=4x+1$
4. Add 11 to it.	40	$4x+1+11=4x+12$
5. Divide it by 4.	10	$\frac{4x+12}{4}=x+3$
6. Subtract 3 from your answer.	7	$x+3-3=x$
7. Ask your client what his final answer is and reveal his favorite number, that is, his final answer.		7

TABLE 2
Example 2: "I Know What Month You Were Born!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel Was Born in March)	Private Eye's Work (Lyta Sets the Month as x)
1. Think of your birth month (e.g., January = 1, February = 2, March = 3, etc.)	3	x
2. Add 9 to your birth month.	12	$x+9$
3. Multiply it by 5.	60	$(x+9)\times 5$ $=5x+45$
4. Subtract 45 from your answer.	15	$5x+45-45$ $=5x$
5. Divide it by 5.	3	$\frac{5x}{5}=x$
6. Ask your client what his final answer is and reveal his birth month, that is, his final answer.		3 March

Type A: Client's final answer = client's secret information (examples 1–3)

Example 1: "I know your favorite number!" Everyone has something favorite. How can we know each individual's favorite number? (See the directions in **table 1**.) When Lyta, the private eye, asked Joel, the client, for the final answer, he gave his favorite number.

Example 2: "I know what month you were born!" The directions in **table 2** are designed to end with the original starting number that indicates the person's secret birth month (1 = January, 2 = February, and so on).

Example 3: "I know your mother's age!" Since a mother's age is two digits, the directions in **table 3** require two variables: the tens place digit and the ones place digit. Joel's final answer directly indicated his mother's age, since the final form, $10x + y$, is the expanded form of two-digit numbers.

TABLE 3
Example 3: "I Know Your Mother's Age!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel's Mother is 39 Years Old)	Private Eye's Work (Lyta Sets the 10's Place Digit as x; the 1's Place Digit as y)
1. Think of the 10's place digit of your mother's age.	3	x
2. Multiply it by 5.	15	$5x$
3. Add the 1's place digit of your mother's age to it.	24	$5x+y$
4. Multiply it by 2.	48	$2(5x+y)=10x+2y$
5. Subtract the 1's place digit of your mother's age from it.	39	$10x+2y-y$ $=10x+y$
6. Ask your client what his final answer is and reveal his mother's age, that is, his final answer.		39

TABLE 4
Example 4: "I Know Your Favorite Day of the Week!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel's Favorite Day Is Sunday)	Private Eye's Work (Lyta Sets the Day as x)
1. Think of your favorite day of the week (e.g., Sunday = 1, Monday = 2, Tuesday = 3, etc.)	1	x
2. Multiply it by 5.	5	$5x$
3. Add 20 to it.	25	$5x+20$
4. Multiply it by 2.	50	$2(5x+20)$ $=10x+40$
5. Subtract 30 from it.	20	$10x+40-30$ $=10x+10$
6. Ask the client what number he got so far.		$10x+10=20$
7. Take away 10 from it.		$20-10=10$
8. Divide it by 10.		$\frac{10}{10}=1$
9. Reveal the client's favorite day based on the answer in step 8.		Sunday

$10x+10-$ $=10x$ $\frac{10x}{10}=x$

Type B: Client's final answer = clue for the secret information (examples 4–7)

In the following examples, the shaded section of each table represents the private eye's extra work..

Example 4: "I know your favorite day of the week!" In this example, the client's final answer does not provide the client's favorite day of the week. (See **table 4**.) To find this information, Lyta, the private eye, did more work by subtracting 10 from the client's final answer and dividing it by 10.

Example 5: "I know what numbers you are thinking of! Three at a time!" Although this example asks for three numbers, the private eye needs only one variable since the numbers are consecutive (see

TABLE 5

Example 5: "I Know What Numbers You Are Thinking Of! Three at a Time!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel's Choices Are 7, 8, and 9)	Private Eye's Work (Lyta Sets the Three Numbers as $x - 1$, x, and $x + 1$)
1. Think of three consecutive numbers.	7, 8, 9	x-1, x, x+1
2. Add them all.	24	x-1+x+x+1 = 3x
3. Ask your client what number he got.		3x=24
4. Divide it by 3.		24/3 = 8 3x/3 = x
5. Find the previous number and next number by subtracting and adding 1.		7, 8, 9 x-1, x, x+1
6. Reveal the three numbers in step 5.		7, 8, 9

TABLE 6

Example 6: "I Know What Your Birthday Is!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (Joel's Birthday Is March 22)	Private Eye's Work (Lyta Sets the Month as x and the Date as y)
1. Multiply your birth month by 60.	180	60x
2. Add your birth date to it.	202	60x+y
3. Tell me what you got so far.		60x+y = 202
4. Ask your client to start over: Multiply the month by 60.	180	60x
5. Subtract your birth date from it.	158	60x − y
6. Tell me what you got.		158
7. Solve the simultaneous equations using the results from steps 3 and 6.		{60x+y=202 " A {60x−y=158 " B A + B + 60x+y=202 C → A 60x−y=158 y=22 120x=360 x=3 " C
8. Reveal your client's birthday based on the result in step 7.		March 22

table 5). When Lyta asked for the final answer, Joel provided a clue for the middle number, x. Using this clue, Lyta divided Joel's final answer by 3. Then she easily figured out that it was the middle number. The first and the last numbers were found by subtracting and adding 1 to it.

Example 6: "I know what your birthday is!" This example requires solving simultaneous equations (see **table 6**). Lyta asked for two values ($60x + y$ and $60x - y$) to solve the simultaneous equations. By knowing two values, she found out Joel's birthday.

Example 7: "I know the last three digits of your telephone number!" This is another example of finding three numbers at a time; however, the numbers are not consecutive. The directions in **table 7** were not complicated for Joel, because he was simply

TABLE 7

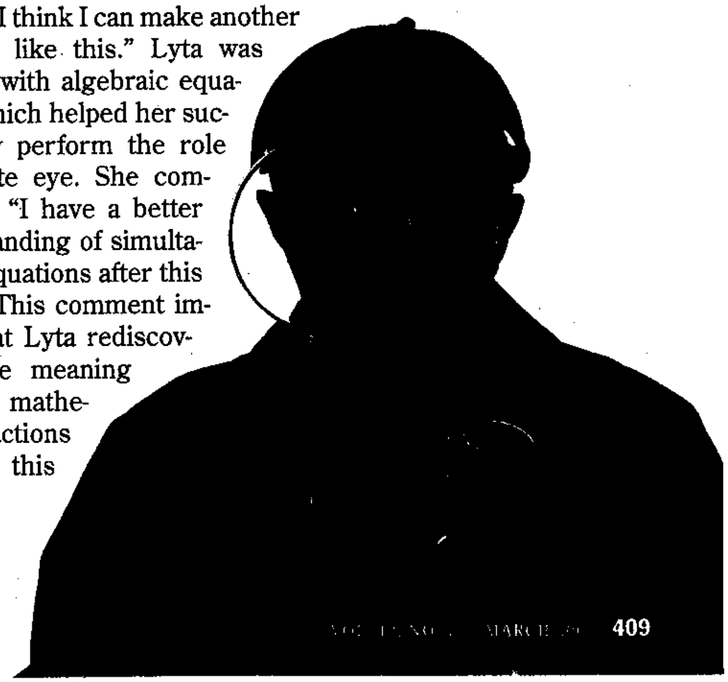
Example 7: "I Know the Last Three Digits of Your Telephone Number!"

Directions for the Private Eye (Ask Your Client the Following Questions, Step by Step)	Client's Work (The Last Three Digits of Joel's Phone Number Are 4, 2, and 9)	Private Eye's Work (Lyta Sets the Three Digits as A, B, and C)
1. Add the first two numbers and tell me what you got.	6	XXX-XABC A+B=6
2. Add the last two numbers and tell me what you got.	11	B+C=11
3. Add the first and last numbers and tell me what you got.	13	A+C=13
4. Solve the simultaneous equations.		{A+B=6 " a {B+C=11 " b {A+C=13 " c a − b − A+B=6 B+C=11 A−C=−5 " C c + b + A+C=13 → a A−C=−5 → b 2A=8 A=4 " d ∴429
5. Reveal the numbers.		429

asked to provide the sum of two single-digit numbers three times. However, Lyta had to do extra work to reveal the last three digits of Joel's phone number.

Reflections

THE STUDENTS' DISCUSSION AFTER WORKING through the examples described above showed their interest in doing each activity, reasoning how it works, and creating their own versions of games. Joel was more interested in type B problems. In example 7, he made some conjectures on the shaded directions, listing the possible pairs of numbers in each case. Lyta and Joel had an extensive discussion on this approach. **Table 8** shows the summary of their discussion. It was not the procedure we originally intended, but it worked. Joel also mentioned, "I think I can make another problem like this." Lyta was familiar with algebraic equations, which helped her successfully perform the role of private eye. She commented, "I have a better understanding of simultaneous equations after this game." This comment implies that Lyta rediscovered the meaning of her mathematical actions through this activity.



TRANSFORMATIONS IN MOTION

This activity will be done as a large group under teacher direction.

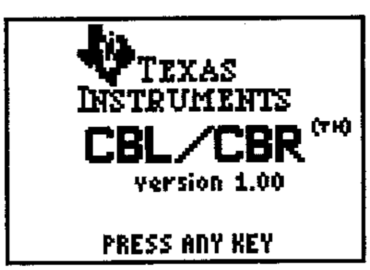
Materials: 2 CBRs, one labeled function f and the other labeled function g, 2 Calculators with CBR/CBL application, link cable, TI presenter or graph link for computer projection or TI SmartView

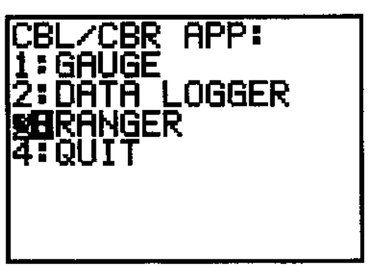
Enlist the assistance of 2 students to hold the linked calculators and CBRs and walk to collect data.

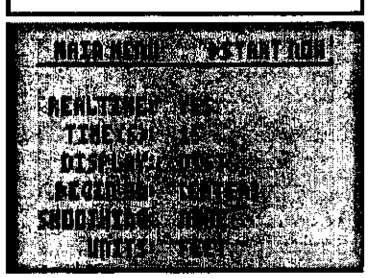
Procedure: One of the data collectors is to be assigned the label of <u>Function F.</u> The data collected by this student represents f(x).
The other data collector is to be assigned the label of <u>Function G</u>. The data collected by this student represents g(x). We will, by the end of this lesson, write g(x) in terms of a transformation on f(x).

The set up:

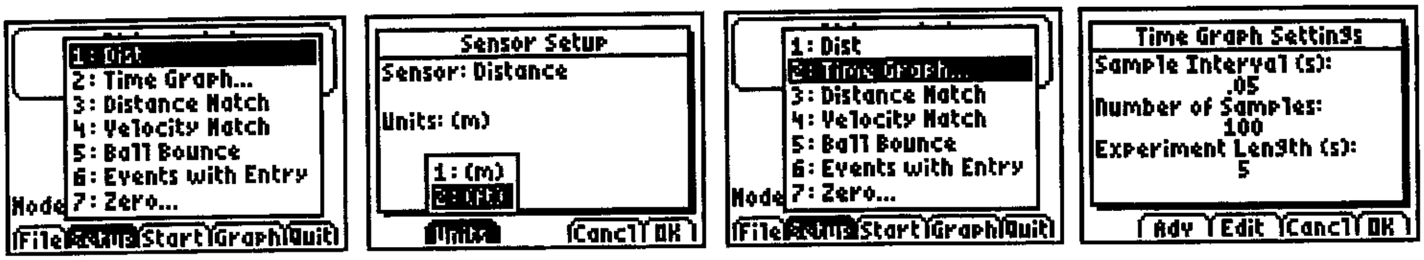
CBR1s must have the CBR application installed. Link the CBR to the calculator. Press APPs, CBR/CBL, Ranger, Set up You want real time and feet. Then keep the calculator linked. (You could select real time, trigger, and feet so the calculator would not need to remain connected to the CBR during the data collection. However, students would not be able to see the path they walk. Just make sure that the CBR does not have low battery power or else the trigger might not work correctly.)
CBR2s must have the EasyData application installed.







Using the CBR2: Link the CBR and calculator and select the EasyData Application. The EasyData Time Graph Set Up automatically is set for 5 seconds. If students want more (or less) time, they need to either change the time interval or change the number of samples (i.e., changing the Number of Samples to 200 would give the group 10 seconds of data).

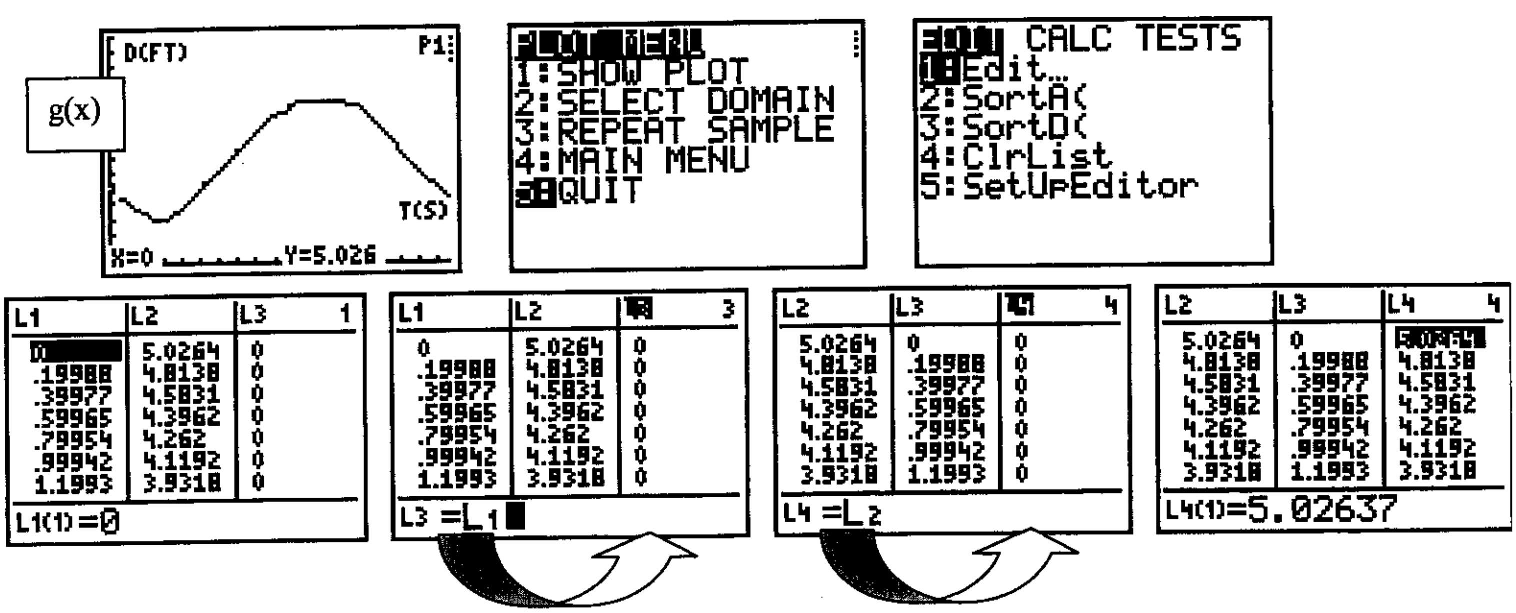


The activity:

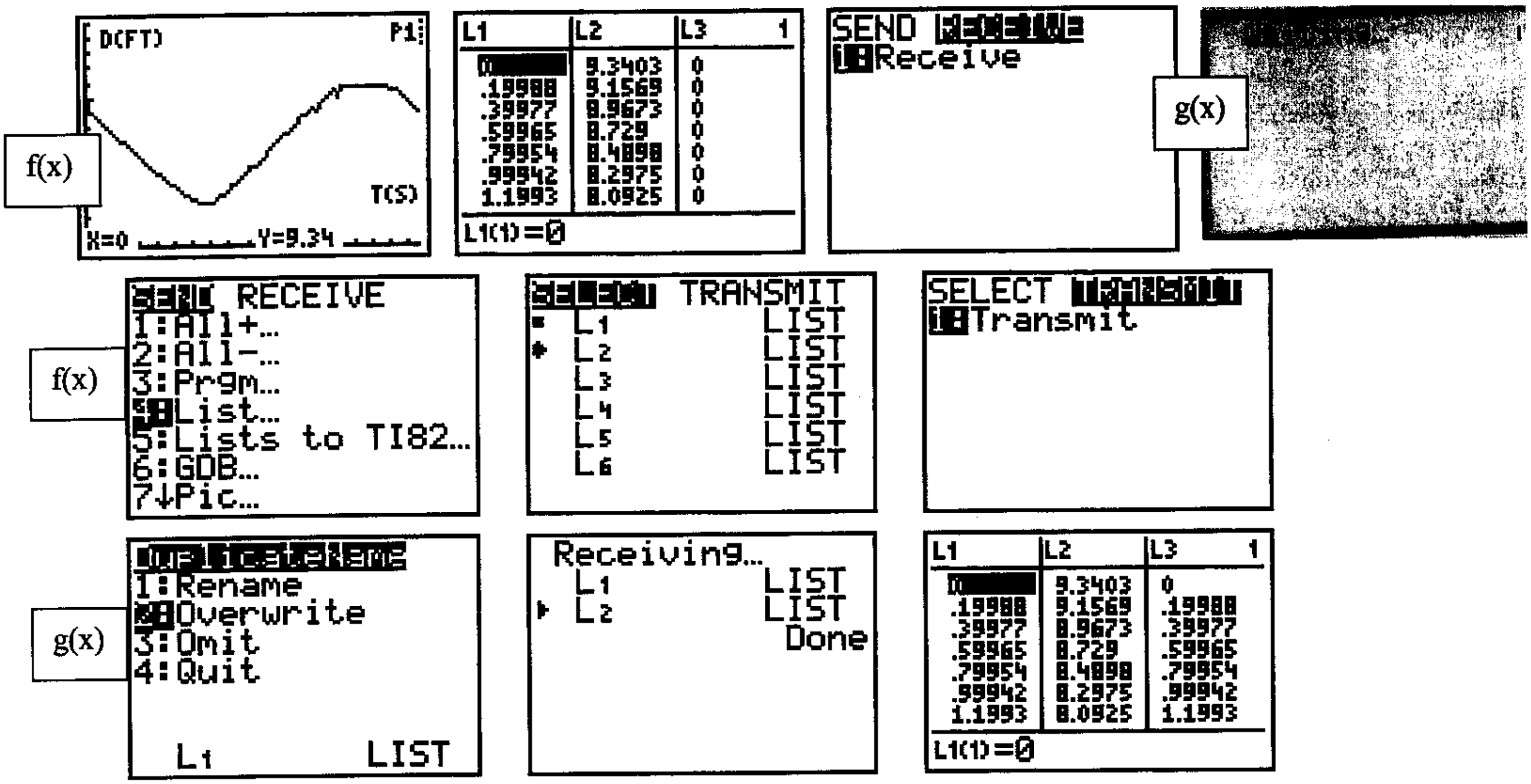
Have the 2 walkers link arms at elbow and practice walking slowly together.
When they have the simultaneous stepping aligned, have the students hold the CBRs while facing the wall or door. The CBRs need to be aligned, as well. The best data comes from using the smoothest surface to walk toward.
Both students start walking toward the wall. The student designated as function f presses enter and starts collecting data immediately.
The student designated as function g counts to 3 and then presses enter to collect data
Both students continue to walk toward wall, walk backward, then either stand still or walk toward wall again until both CBRs stop collecting data.

The analysis:

The student designated as function g needs to enter and quit the program, go to STAT, copy L_1 into L_3 and L_2 into L_4. (A sample of data is used here.)

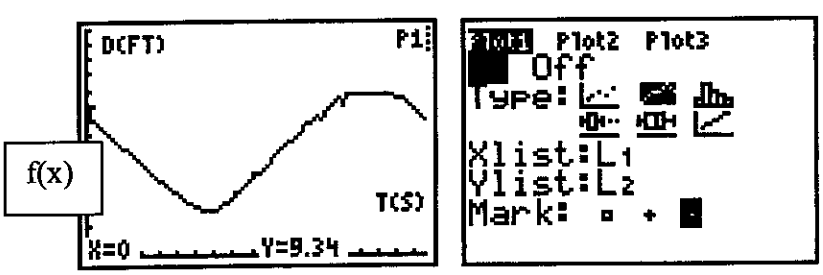


Then link calculator of student designated as function f to calculator with function g. Set function g calculator to receive. Set function f to transfer L_1 and L_2. <u>Overwrite</u> old L_1 and L_2.

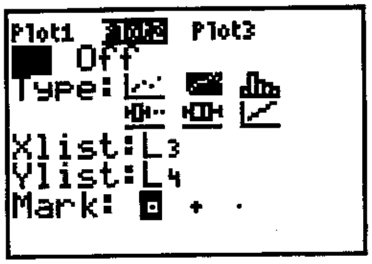


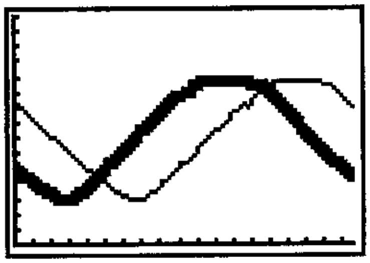
Have the students note that the time (independent variable) listed in L_1 and L_3 for both functions is the same. The calculators collected the same amount of data at the same intervals for both functions. Have the students note that the distance from the wall (dependent variable) is not the same. Ask students "Why not?" Also, have students compare the meaning of the y-intercept for each function.

Attach function g calculator to the TI- presenter or computer. Graph. Only function f should appear because statplot 1 is on with L_1 and L_2 as the sets of points. Discuss with students y-intercept, direction of line, (perhaps slope of line), shape of line, distance from wall, etc.



Then turn on statplot 2, connected graph, xlist: L_3 and ylist: L_4 and graph. To make this graph come up darker be sure to have the open square marked. Now, everyone should see that g(x) is a transformation of the first graph b(x).





Emphasize (the point of this activity):

Even though we started function g 3 seconds <u>later</u> than function f, function g is a translation 3 units <u>to the left</u> of function f.

Emphasize the graph! Look at and compare the significant points on the graphs. Show that the graph translated to the left 3 units by starting 3 seconds later. We want students to know that we would need to add those 3 seconds to any of the x-coordinates for function f to get function g to move on top of function f. In that case, g(x) = f(x + 3). Students need to realize that the expression (x + 3) refers to something happening later since we have to add time to the x-coordinate and that "later" means a translation to the left.

Repeat activity:
Follow the procedure again, this time, however, with the student designated as function g starting 3 seconds before student designated as function f.

Collect and transfer the data in the same manner. Analyze the graphs in the same manner.

Emphasize:

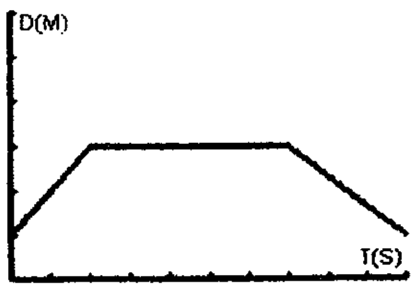
Even though we started function g 3 seconds <u>sooner</u> than function f, function g is a translation 3 units <u>to the right</u> of function f.

When you compare the graphs of f(x) and g(x), g(x) is a translation to the right of f(x). To match the y-coordinates, we would need to subtract 3 seconds from each of the x-coordinates of function f to get function g to move on top of function f. Therefore, g(x) = f(x – 3). They need to realize that the expression (x – 3) refers to something happening sooner since we have to subtract time to the x-coordinate and that "sooner" means a translation to the right.

Walking Piecewise Graphs

a) Write a description for a walk which could produce each of these graphs.
b) Walk the following linear graphs using a motion detector and a graphing calculator.
c) Write a piecewise function for the distance in terms of the time.
d) Graph the speed in meters per second in terms of the time for each walk.
e) Write a piecewise function for the speed in terms of the time.

1.



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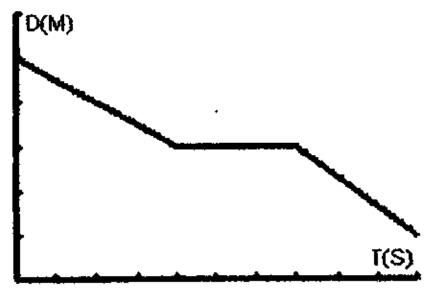

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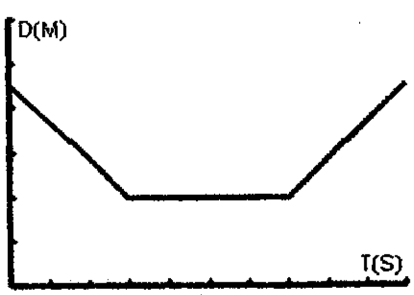

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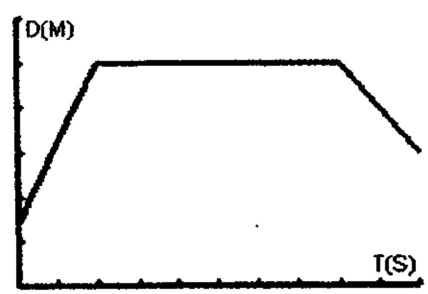
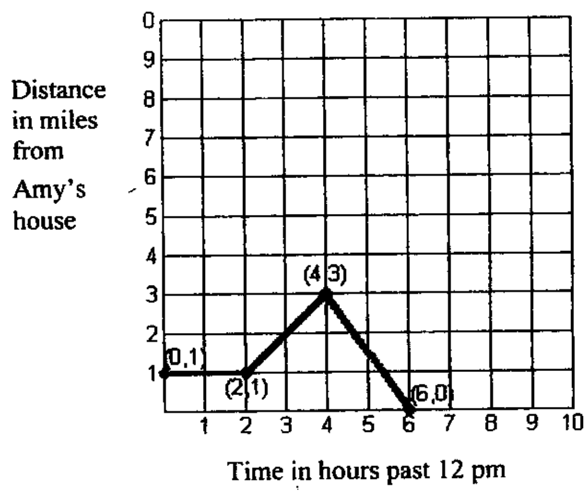


A Transformation Story

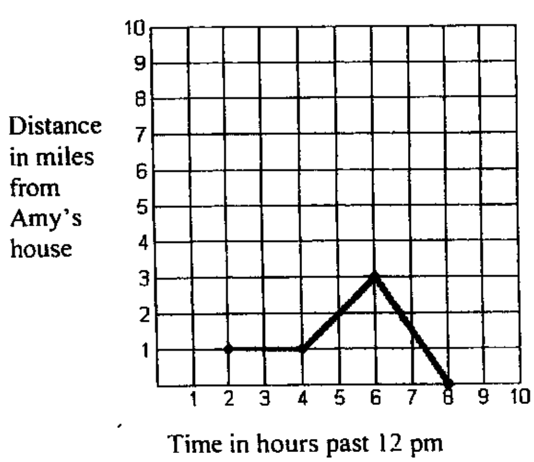
1. Amy was at Julia's house from 12 pm to 2 pm. She went to Jordan's house but he was not home. Jordan lives two miles from Julia's house. Amy arrived home at 6 pm. The following graph describes Amy's path.



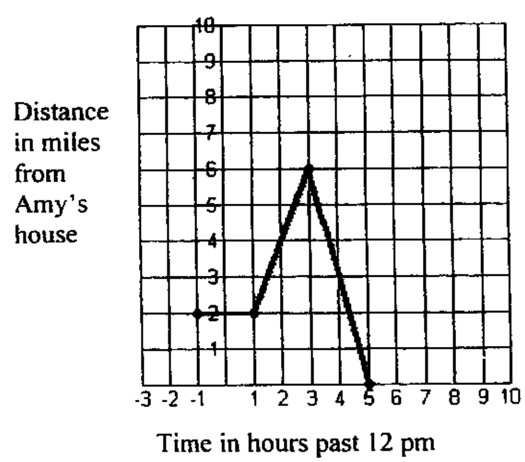
Each of the following graphs describes a change in the problem situation.

Rewrite the story that each graph describes.

a)

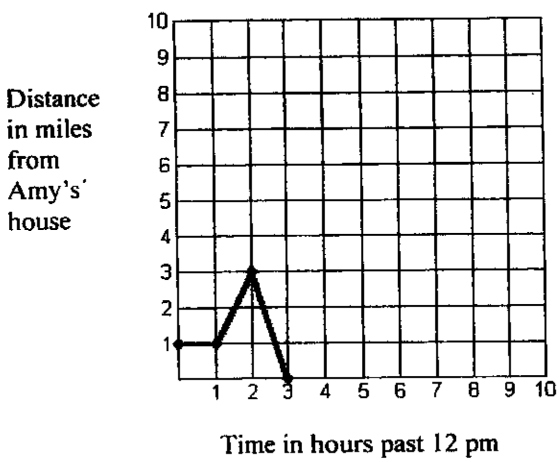


b)

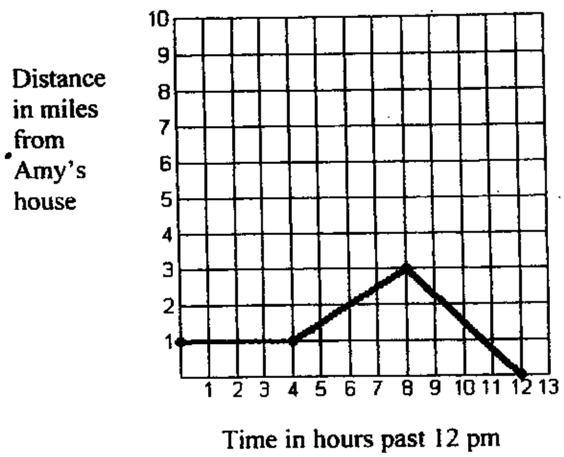


149

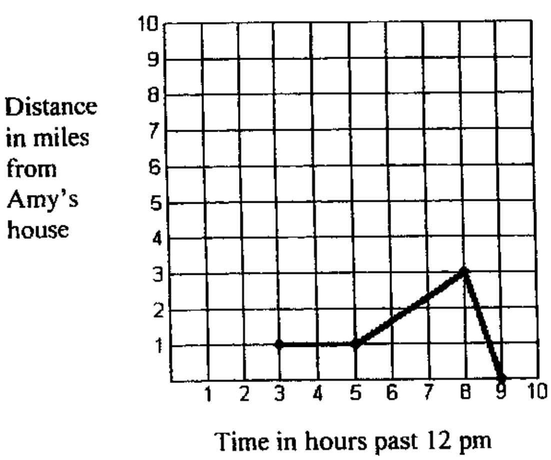
c)


Distance
in miles
from
Amy's
house
Time in hours past 12 pm

d)


Distance
in miles
from
Amy's
house
Time in hours past 12 pm

e)


Distance
in miles
from
Amy's
house
Time in hours past 12 pm

150

f) Which graph describes the following changes from the original problem?

1. Amy left Julia's house at 2 pm.

2. Amy's trip took half the time.

3. Amy stayed at Julia's house from 11 am to 1 pm.

4. Julia's house is twice as far from Amy's.

5. Amy traveled more slowly on her way home.

6. Amy was at Jordan's house at 2 pm.

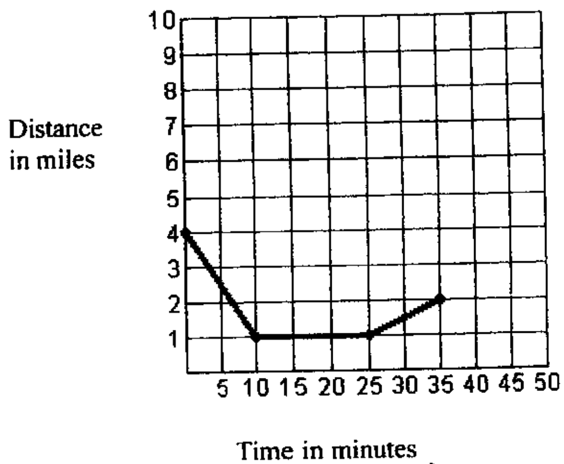
7. Amy traveled to Jordan's house in 4 hours.

8. Amy was traveling the slowest.

9. Amy's speed is 2 mph.

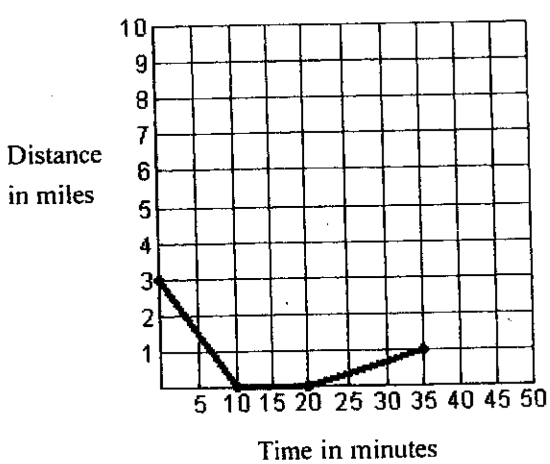
10. The total distance Amy traveled is 12 miles.

2. Write a story to describe the graph below.

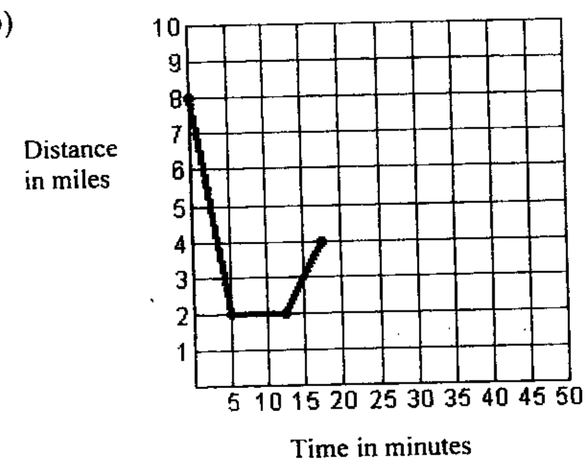


Each graph below describes a change in the story, explain the change.

a)



b)



3. If (2,3) is a point on $f(x)$, what will the coordinates become on the following transformations?

   a) $2f(x)$

   b) $f(x)+1$

   c) $-f(x)$

   d) $f(3x)$

   e) $f(-x)$

   f) $f(x-1)$

   g) $f(x+2)$

   h) $f(x)-3$

   i) $f(\frac{1}{2}x)$

   j) $f(x+2)-1$

4. If $f(x)$ contains the point A(-2, 5), what is a new function if A is transformed to the following point?

   a) (-1, 5)

   b) (3, 6)

   c) (-4, 5)

5. If $f(x)$ represents the amount of money Molly made and will make in hundreds of dollars from Wednesday of one week to Thursday of the next week, explain the meaning of $f(x\text{-}1)$ in terms of this situation?

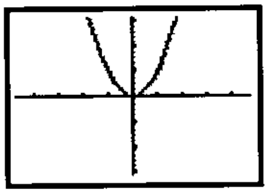
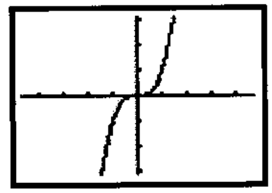
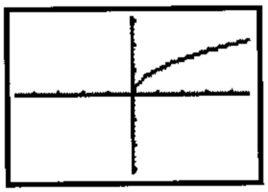
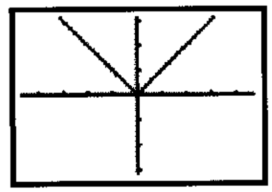
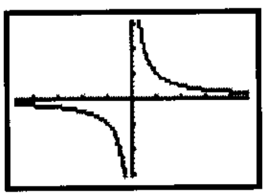
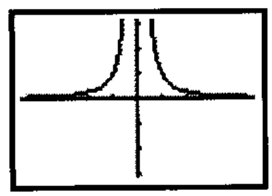
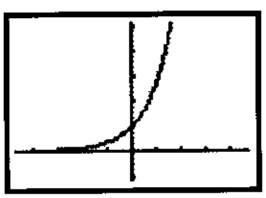
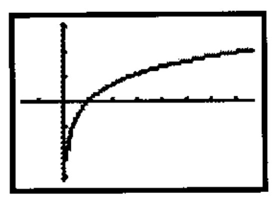
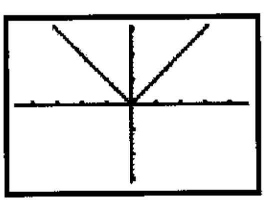
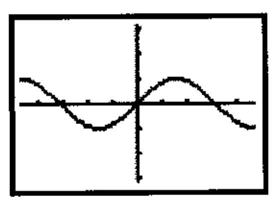
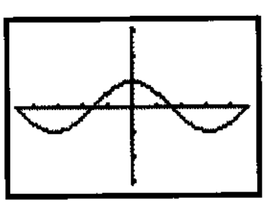
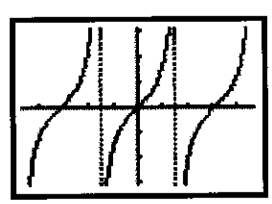
Function Rule Cards

$y = x^2$	$y = x^3$
$y = \sqrt{x}$	$y = \sqrt{x^2}$
$y = \dfrac{1}{x}$	$y = \dfrac{1}{x^2}$
$y = e^x$	$y = \ln x$
$y = \lvert x \rvert$	$y = \sin x$
$y = \cos x$	$y = \tan x$

Function Type Cards

quadratic function	cubic function
radical function	radical function
rational function	rational function
exponential function	logarithmic function
absolute value function	trigonometric function
trigonometric function	trigonometric function

Graph Cards

Domain Cards

Domain: {all real numbers}	Domain: {all real numbers}
Domain: $\{x \geq 0\}$	Domain: {all real numbers}
Domain: {all real numbers except 0}	Domain: {all real numbers except 0}
Domain: {all real numbers}	Domain $\{x > 0\}$
Domain: {all real numbers}	Domain: {all real numbers}
Domain: {all real numbers}	Domain: {all real numbers except $x = \frac{\pi}{2} + k\pi$ where k is an integer}

Domain Cards

Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$
Domain: $[0, \infty)$	Domain: $(-\infty, \infty)$
Domain: $(-\infty, 0) \cup (0, \infty)$	Domain: $(-\infty, 0) \cup (0, \infty)$
Domain: $(-\infty, \infty)$	Domain: $(0, \infty)$
Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$
Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty) \cap \left(x \neq \frac{\pi}{2} + k\pi\right)$ for all $k \in \mathrm{I}$

Range Cards

Range: $\{y \geq 0\}$	Range: {all real numbers}
Range: $\{y \geq 0\}$	Range: $\{y \geq 0\}$
Range: {all real numbers except 0}	Range: $\{y > 0\}$
Range: $\{y > 0\}$	Range: {all real numbers}
Range: $\{y \geq 0\}$	Range: $\{-1 \leq y \leq 1\}$
Range: $\{-1 \leq y \leq 1\}$	Range: {all real numbers}

Domains, Ranges, and Graphs of Parent Functions

Sketch the graph and identify the domain and range for each of the given parent functions.

Function Type	Domain	Range	Graph
$y=\sqrt{x}$			
$y=x^2$			
$y=x^3$			
$y=\sqrt{x^2}$			
$y=\frac{1}{x}$			

Function Type	Domain	Range	Graph
$y = \dfrac{1}{x^2}$			
$y = e^x$			
$y = \ln x$			
$y = \lvert x \rvert$			
$y = \sin x$			
$y = \cos x$			
$y = \tan x$			

AP Calculus AB Name:____________________

Functions Continued

Here are the rest of our "18 Basic Functions." Complete the chart on your own for these functions.

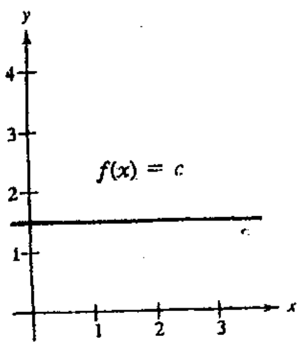
Use notation of choice

Function Type	Domain	Range	Graph
$y = x^{1/3}$			
$y = x^{2/3}$			
$y = [\![x]\!]$			
$y = \cot x$			
$y = \sec x$			
$y = \csc x$			

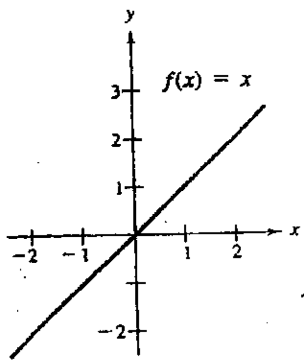
18 Basic Functions Dance

x	$x^{1/3}$	tan x
$\lvert x \rvert$	$x^{2/3}$	cot x
x^2	$\frac{1}{x^2}$	cos x
$\sqrt{x}$	$\lVert x \rVert$	sin x
$\frac{1}{x}$	e^x	sec x
x^3	ln x	csc x

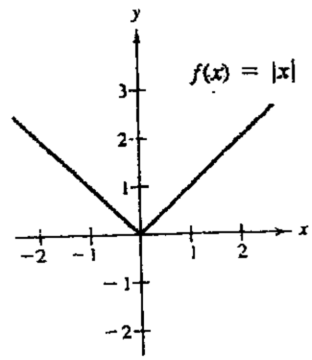
GRAPHS OF COMMON FUNCTIONS PARENT FUNCTIONS



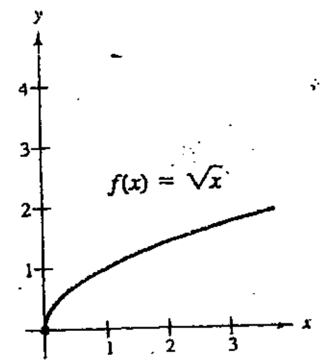
Constant Function



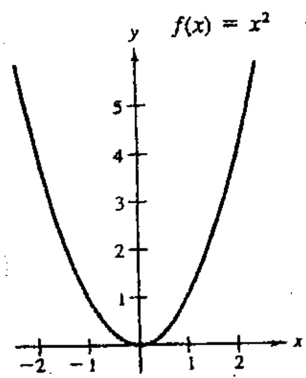
Identity Function



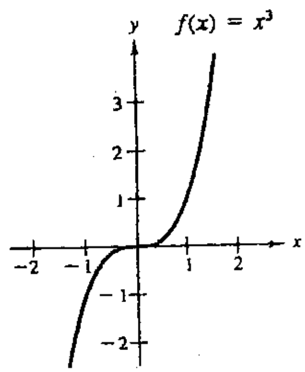
Absolute Value Function



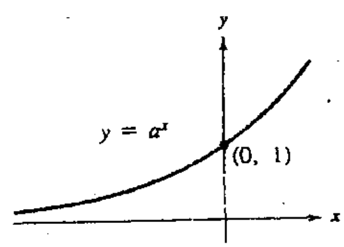
Square Root Function



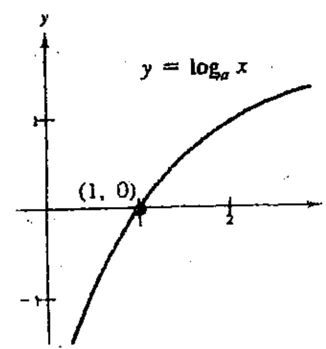
Squaring Function



Cubing Function



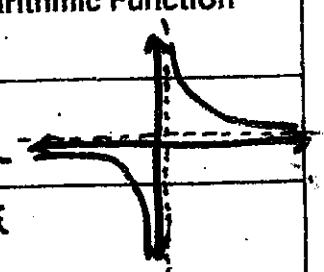
Exponential Function

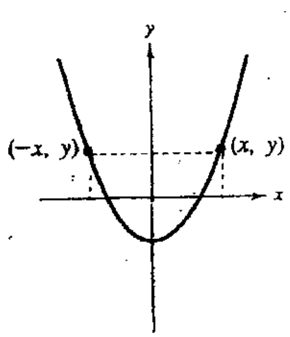


Logarithmic Function

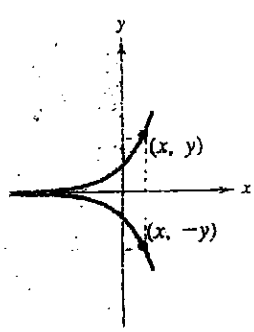
SYMMETRY

Rational Function $f(x) = \frac{1}{x}$

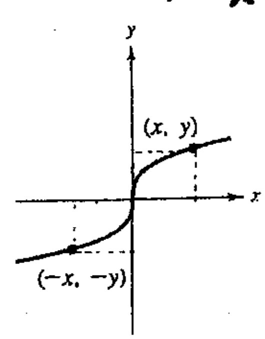




***y*-Axis Symmetry**



***x*-Axis Symmetry**

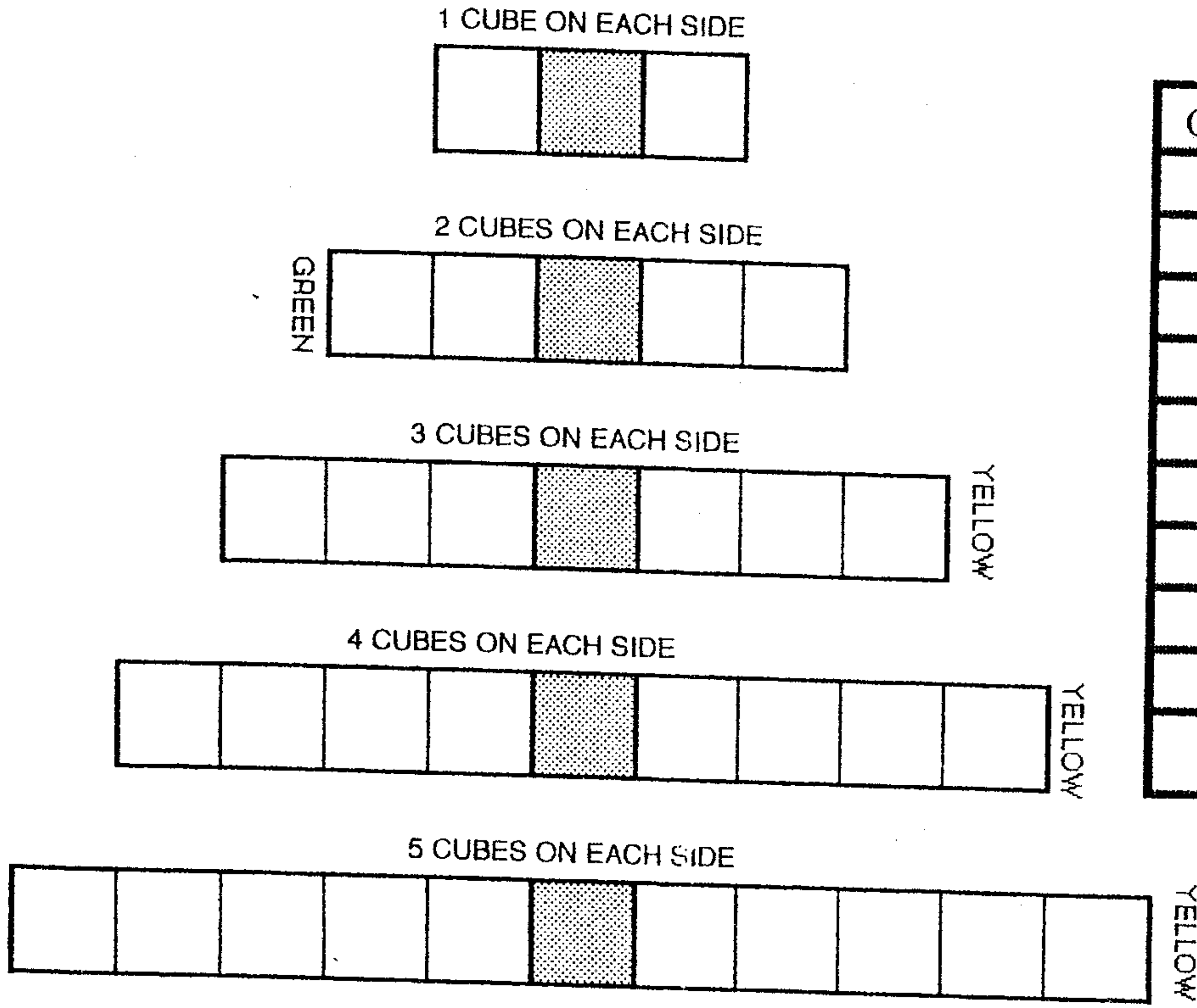


Origin Symmetry

THE SHUTTLE GAME

Start with green cubes on one side and yellow on the other.
The goal is to reverse the colors in the fewest moves possible, following these rules:

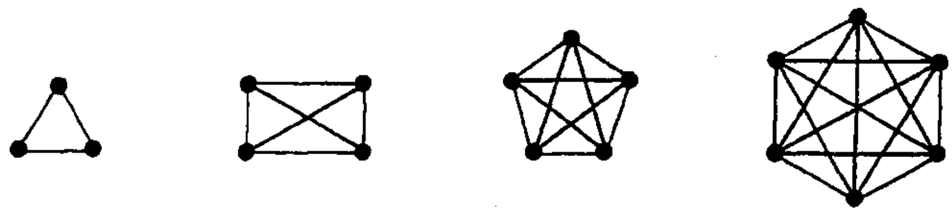
1) A moves means to either move one space or jump one over one tile (opposite color).
2) You may not move backwards.



CUBES	MOVES
1	
2	
3	
4	
5	
6	
7	
10	
50	
N	

DIAGONALS OF A POLYGON

A diagonal of a polygon is a line segment joining any two non-adjacent vertices. Here, n represents the number of sides in the polygon.



$n=3$ $n=4$ $n=5$ $n=6$

In the table to the right, D represents the number of diagonals in a given polygon.

Complete the table at the right through a nine-sided polygon.

Find a pattern which will enable you to find the diagonals in a 50-sided polygon without drawing it.

How did you find the pattern?

Express the general rule for finding the number of diagonals in an n-sided polygon.

n	D
3	____
4	____
5	____
6	____
7	____
8	____
9	____
⋮	
50	____
⋮	
n	____

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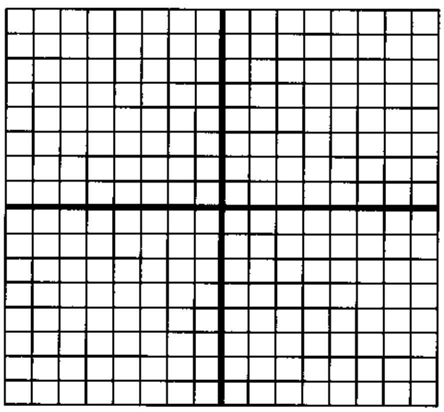
Activity 1: Investigating the Role of *a*

Sketch a graph of the following using a graphing calculator. Your observations should include some table values.

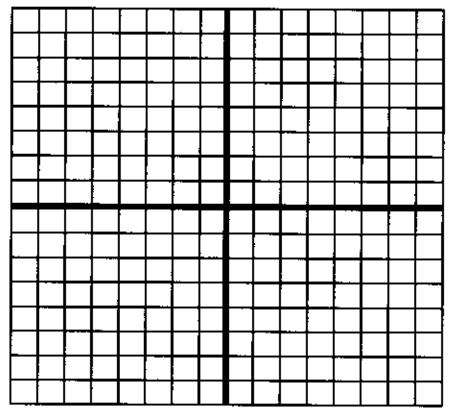
1. Function: $y = x^2$

Observations:



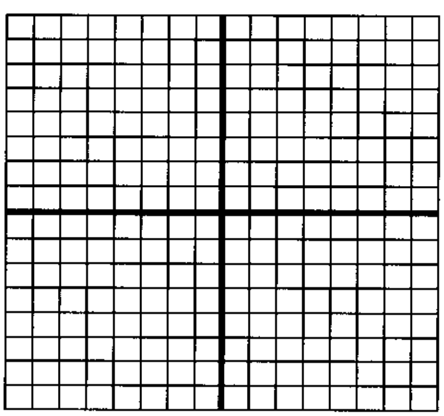
2. $y = 2x^2$, $y = \frac{1}{2}x^2$, $y = 0.2x^2$

Observations:



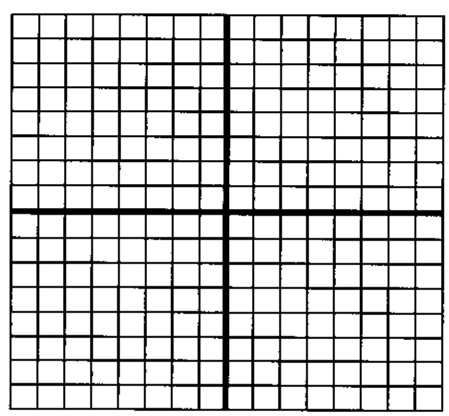
3. $y = 5x^2$, $y = -x^2$

Observations:



4. $y = -\frac{1}{10}x^2, y = -25x^2$

Observations:

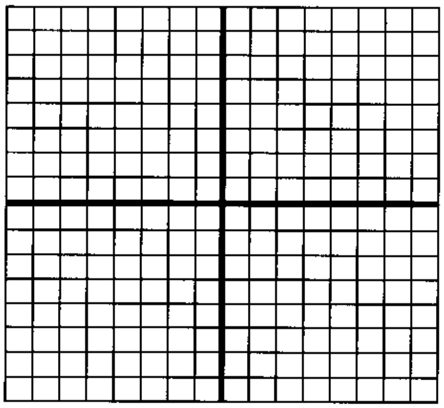


5. In general, what effects do different values of *a* have on the graph of $y = ax^2$?

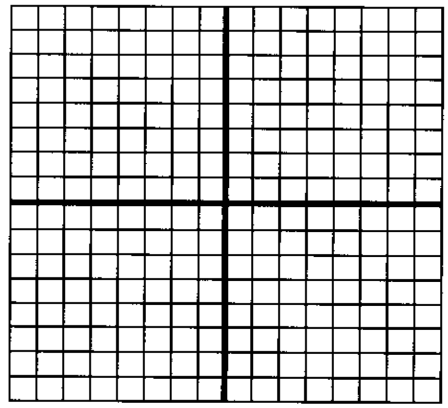
Activity 2: Investigating the Role of *c*

Sketch a graph of the following using a graphing calculator. Your observations should include some table values.

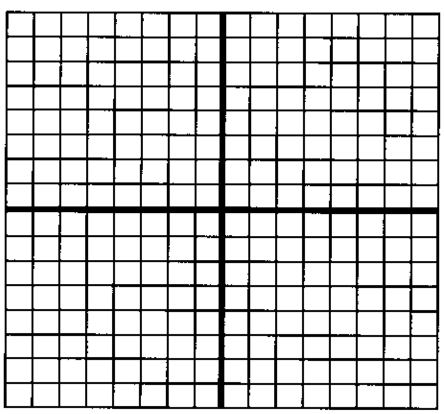
1. $y = x^2$, $y = x^2 + 2$, $y = x^2 + 3$
   Observations:



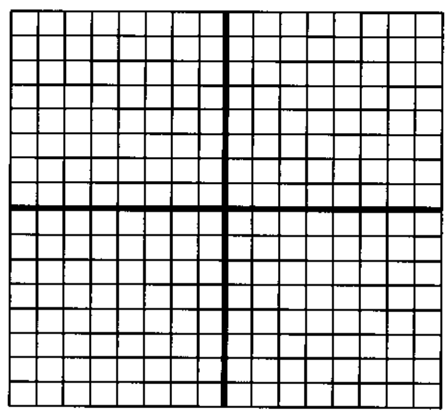
2. $y = x^2 - 0.5$, $y = x^2 - 1$
   Observations:



3. $y = x^2 + 1.5$, $y = x^2 - 2.5$
   Observations:



4. $y = x^2 + 15$, $y = x^2 - 200$
   Observations:



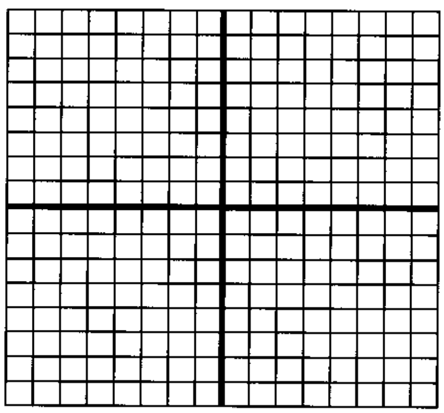
5. In general, what effects do different values of c have on the graph of $y = x^2 + c$?

Activity 3: Investigating the Role of *h*

Sketch a graph of the following using a graphing calculator. Your observations should include some table values.

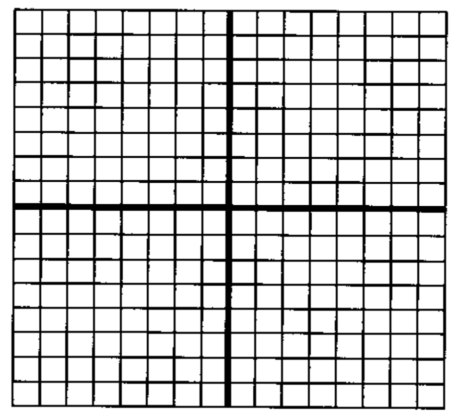
1. $y = x^2$, $y = (x+2)^2$, $y = (x+1)^2$

   Observations:



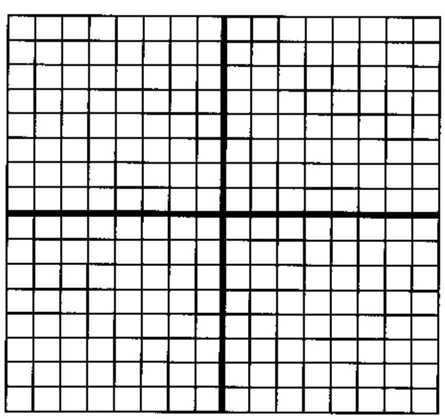
2. $y = (x-1)^2$, $y = \left(x - \frac{1}{2}\right)^2$

   Observations:



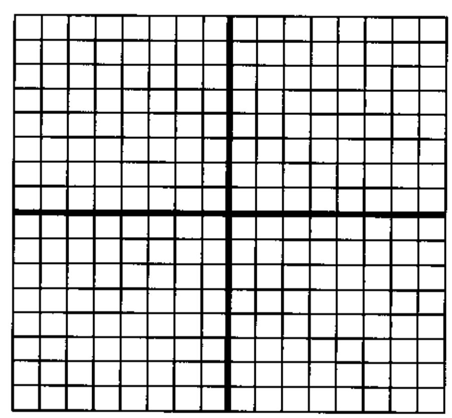
3. $y = (x-3)^2$, $y = (x+2)^2$

   Observations:



4. $y = (x+22)^2$, $y = (x-15)^2$

   Observations:



5. In general, what is the effect on the graph of $y = x^2$, when you replace x with $(x+h)$?

Activity 4: Transformations

1. Fill in the blanks.

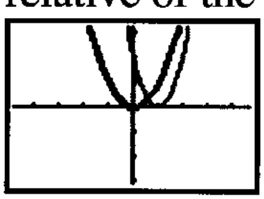
Change from the parent function, $y = x^2$	New Equation	Change in Graph
Add 3 to the function	$y = x^2 + 3$	Vertical translation up 3
		Scale change of 1/3
	$y = 3x^2$	
Replace x with $(x-2)$		
	$y = -x^2$	
		Vertical translation down 2
		Horizontal translation left 1
Multiply by 2		
		Horizontal translation right 3

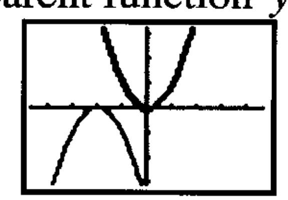
2. Describe the transformations on $y = x^2$ that will produce the graph for each function below. Verify on your calculator.
   a. $y = (x+5)^2 - 1$

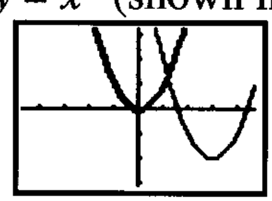
   b. $y = 3x^2 + 2$

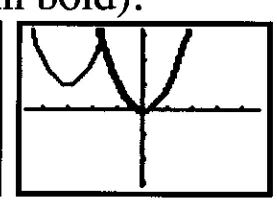
   c. $y = -\frac{1}{3}(x+1)^2$

3. Write an equation for each graph below. Each graph is a relative of the parent function $y = x^2$ (shown in bold).


a.


b.

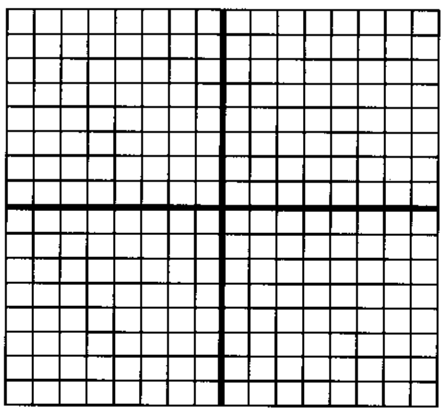

c.


d.

Using your knowledge of transformations on the parent function $y = x^2$, graph the following relatives. Verify with your calculator.

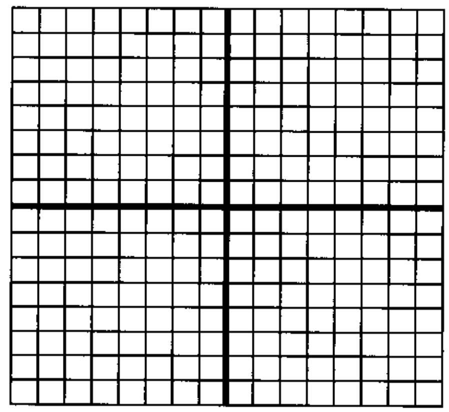
4. Function: $y = 2(x+3)^2$

   Describe transformations:



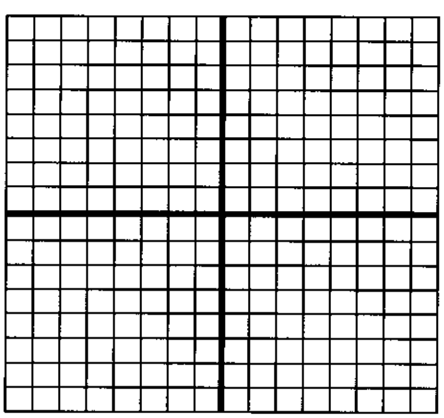
5. Function: $y = \frac{1}{2}(x-2)^2$

   Describe transformations:



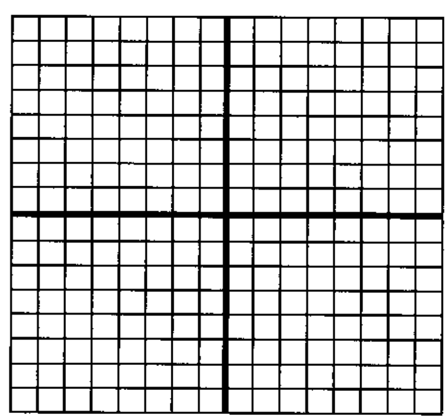
6. Function: $y = -x^2 - 1$

   Describe transformations:



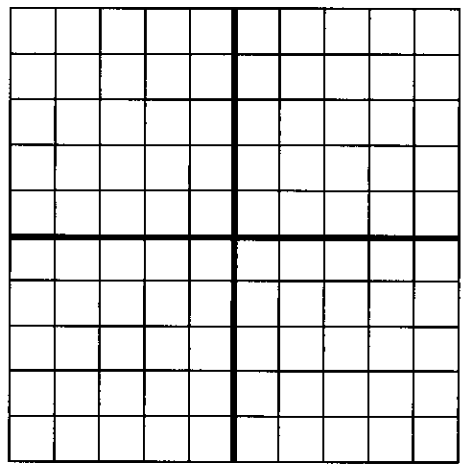
7. Function: $y = (x+2)^2 + 1$

   Describe transformations:



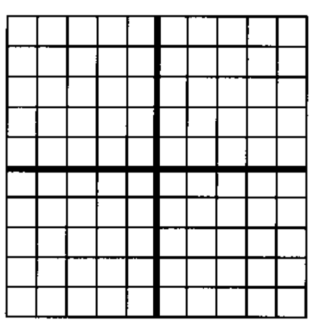
Activity 3: Exploring Horizontal Shifts

1. Describe the following in two ways: $y = x + 4$.

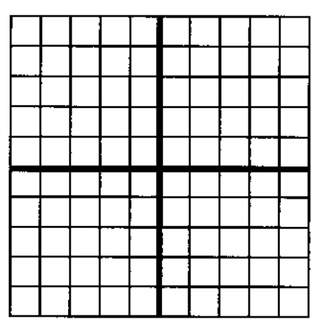


Graph the following lines in two ways. First, as transformations of $y = x$. Then simplify each linear function to $y = mx + b$ or $y = b + mx$ and graph.

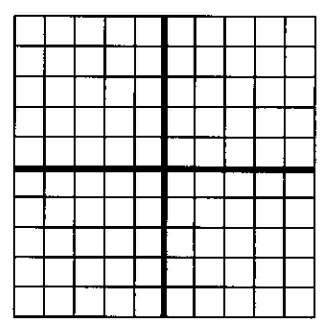
2. $y = (x + 2) + 1$



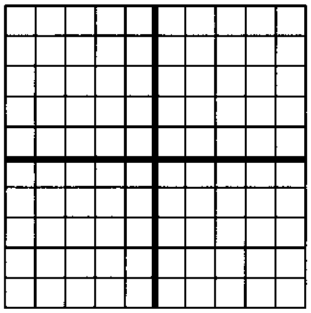
3. $y = (x - 5) + 3$



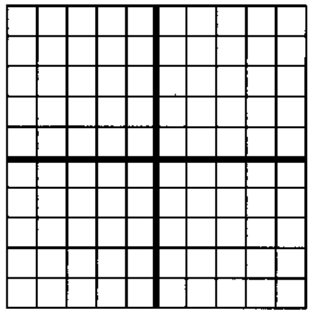
4. $y = (x + 1) - 4$



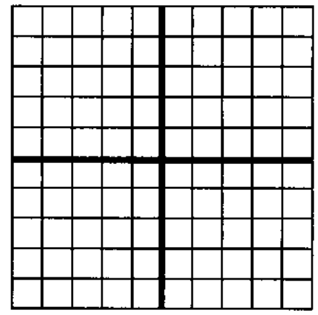
5. $y = 2(x+2)+1$



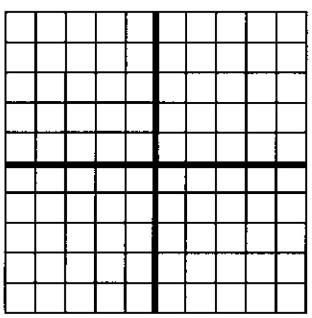
6. $y = \frac{1}{2}(x-4)+3$



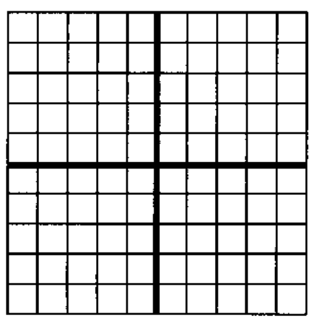
7. $y = 3(x+1)-4$



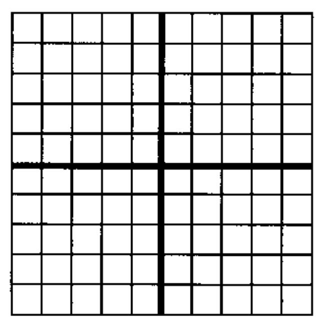
8. $y = -2(x+1)+2$



9. $y = -(x-2)+1$

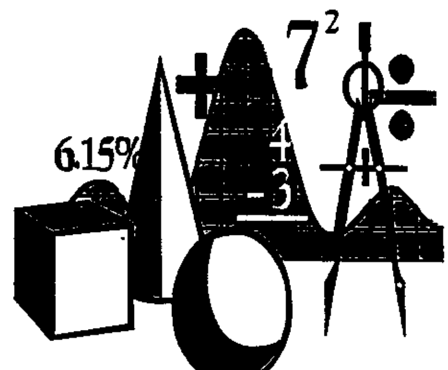


10. $y = -\frac{1}{3}(x+3)$



A Changing Parameters Lesson

from Texas Algebra 1 End-of-Course Training Materials



Changing Parameters

Objective: The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Materials: 1 set of graphs for each group of four
butcher paper
markers
masking tape

Procedure: Divide participants into groups of four. Give each group a set of cards containing graphs. Ask the groups to put their graphs into 3 sets using any criteria they wish. However, each group will be expected to explain their criteria to the other participants. Possibilities might include the following sets.

Cards 1-12: set 1: cards 1, 10, 12 (they are all v-shaped) or (their equations contain $|x|$)
set 2: cards 3, 5, 7, 11 (they are all lines) or (their equations contain x)
set 3: cards 2, 4, 6, 8, 9 (they are all parabolas) or (their equations contain x^2)

or

set 1: cards 1, 2, 4, 6, 10, 12 (they all open up)
set 2: cards 8, 9 (they both open down)
set 3: cards 3, 5, 7, 11 (they are all lines)

or

set 1: cards 1, 3, 4, 8, 11 (they all intersect the origin)
set 2: cards 5, 9, 10 (they all intersect the y-axis above the origin)
set 3: cards 2, 6, 7, 12 (they all intersect the y-axis below the origin)

Participants should be allowed a few minutes to complete this task. Have butcher paper for each group taped to the walls around the room. Then ask a representative from each group to write their sets on the butcher paper using the markers provided. Be sure that these are large enough so that all of the participants can see them. Ask a few of the groups to explain the criteria they used to create their sets. If all of the groups have chosen the same sets, you might suggest some alternatives.

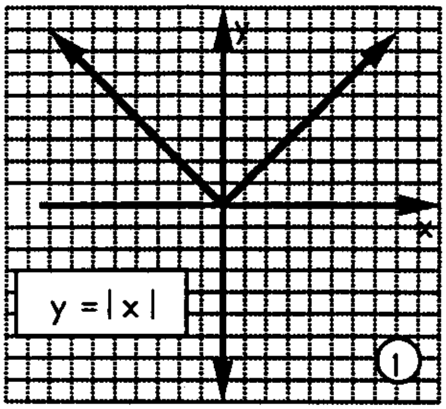
Now ask the groups to divide their cards into two sets. Again, they must be able to explain the criteria they used to form the sets. Possibilities might include the following sets.

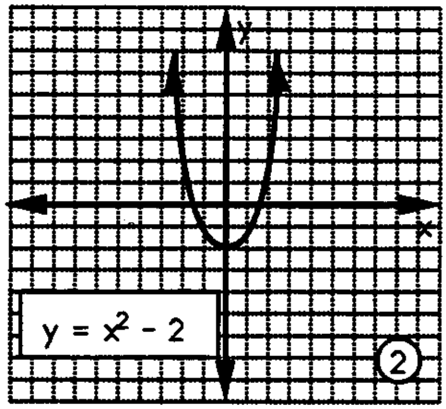
set 1: cards 1,3,4,8,11 (graphs that intersect the x-axis at the origin)
set 2: cards 2,5,6,7,9,10,12 (graphs that do not intersect the x-axis at the origin)

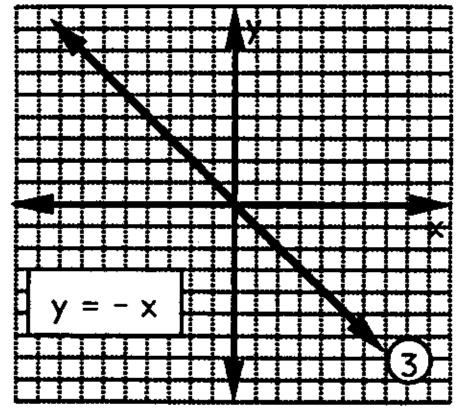
After a few minutes ask the groups to record the sets on the butcher paper. Call on some of the groups to explain their criteria. Once all of the possibilities have been given, concentrate on the two sets above. Discuss the difference and the similarities in the graphs. Also discuss the similarities and differences in the equations that go along with the graphs. Be sure to point out the changes in the graphs and how they relate to the changes in their equations.

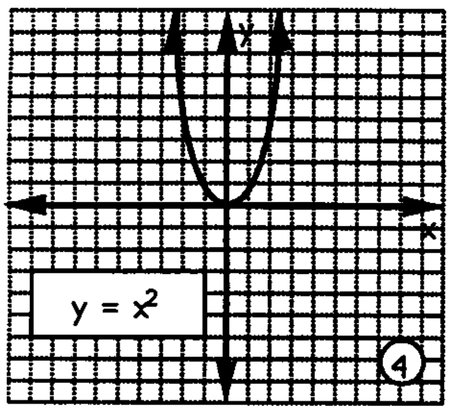
Have participants offer some equations for graphs that will intersect the x-axis above the origin and some that will intersect below the origin.

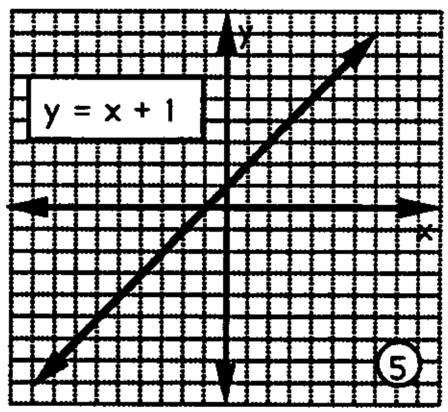
Follow this same activity format for the cards 13 - 36. Cards 13 - 24 are examples of graphs and equations that show movement to the right and to the left. Cards 25 - 36 demonstrate changes in the width of the graphs. Any of the cards could be used to show graphs that open up or down. The sets of cards can be combined in different ways to demonstrate different concepts. The graphs in the "home" position such as $y = x^2$ are repeated in each set of cards.

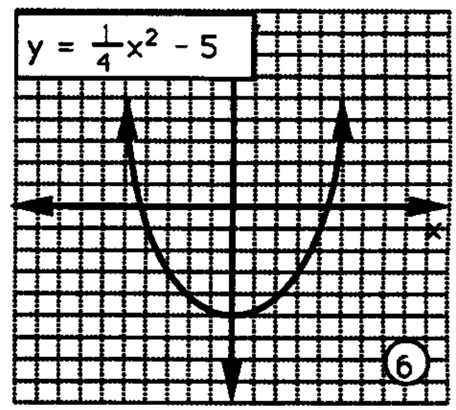

y
x
y = | x |
1

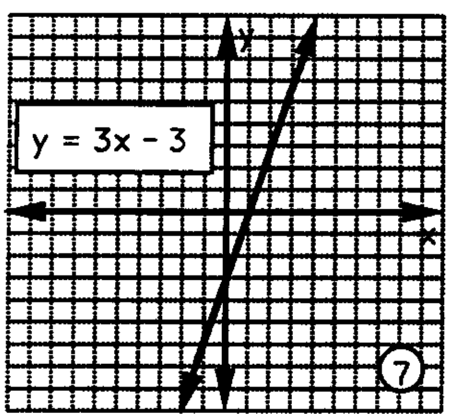

y
x
y = x² - 2
2

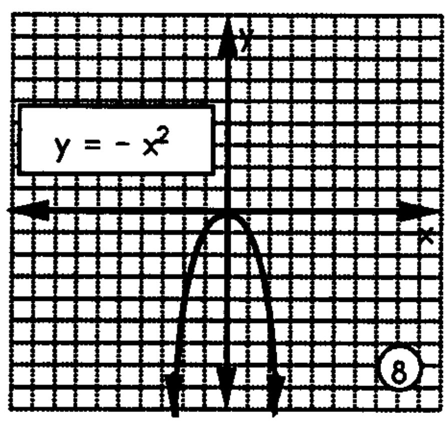

y
x
y = - x
3

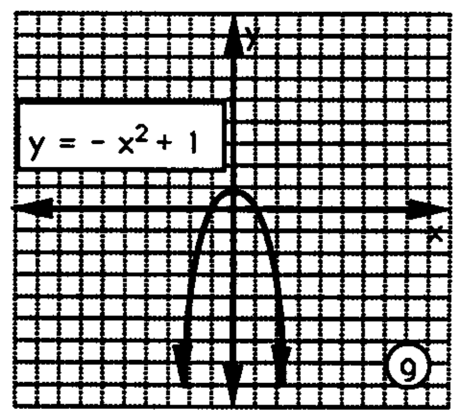

y
x
y = x²
4

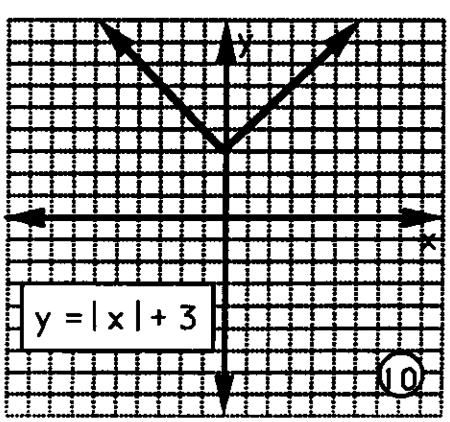

y
x
y = x + 1
5

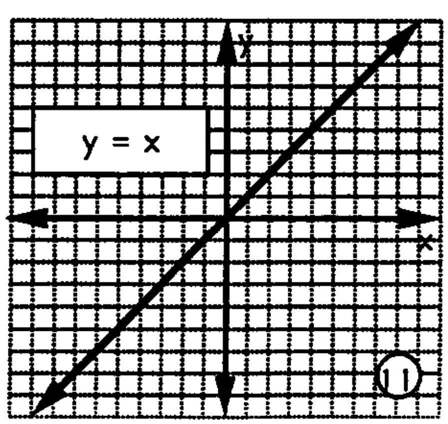

y = 1/4 x² - 5
x
6

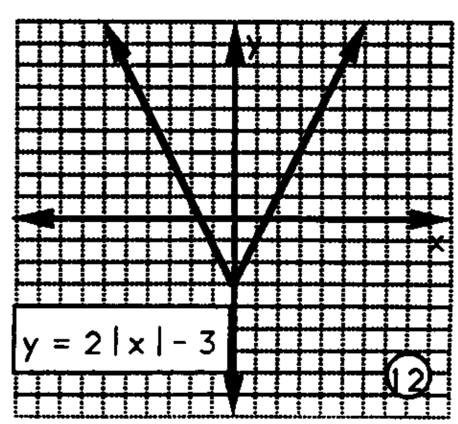

y
x
y = 3x - 3
7

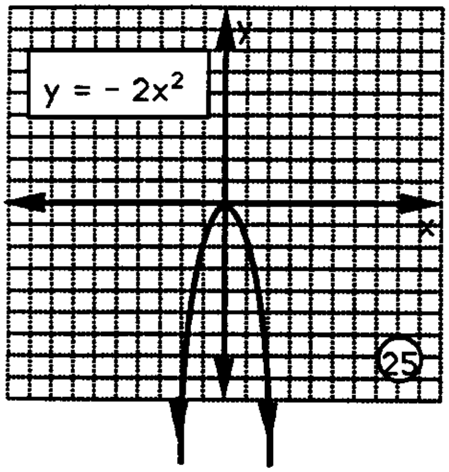

y
x
y = - x²
8

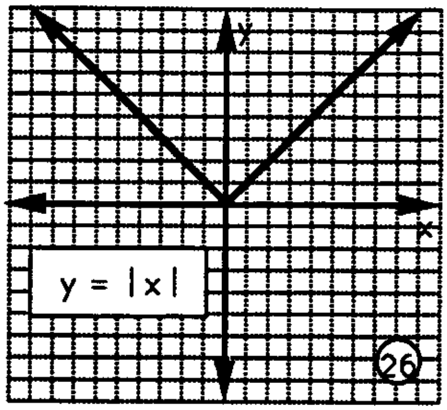

y
x
y = - x² + 1
9

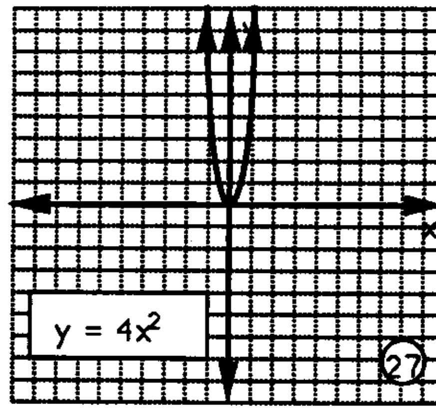

y
x
y = | x | + 3
10

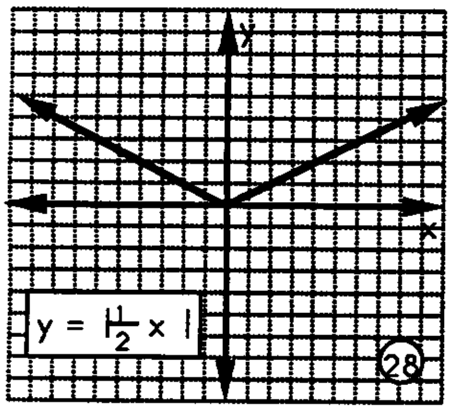

y
x
y = x
11

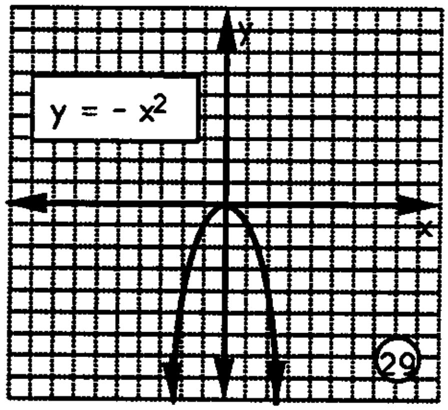

y
x
y = 2 | x | - 3
12

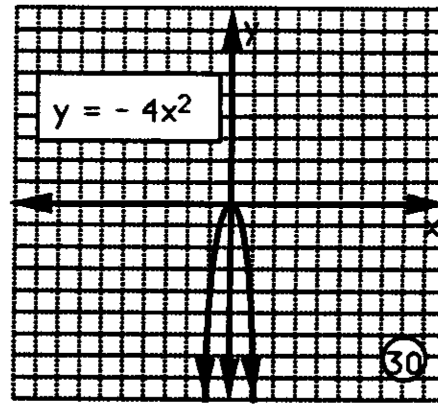

$y = -2x^2$
y
x
25

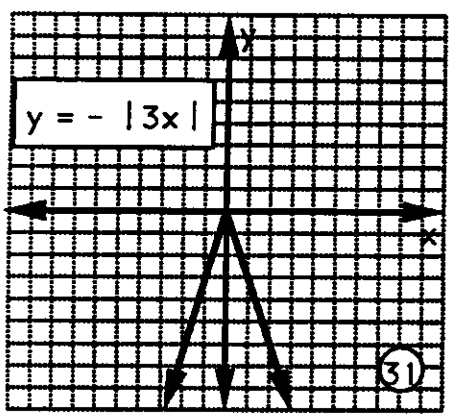

$y = |x|$
y
x
26

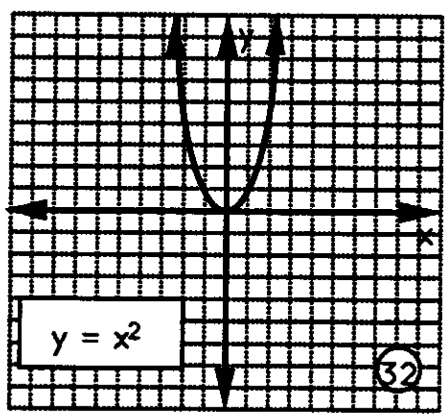

$y = 4x^2$
y
x
27

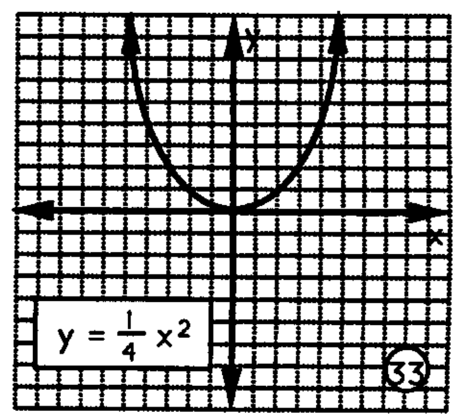

$y = |\frac{1}{2}x|$
y
x
28

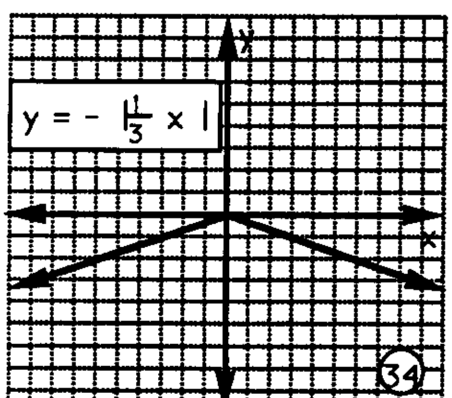

$y = -x^2$
y
x
29

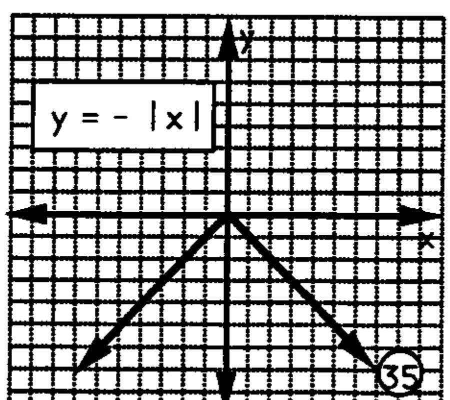

$y = -4x^2$
y
x
30

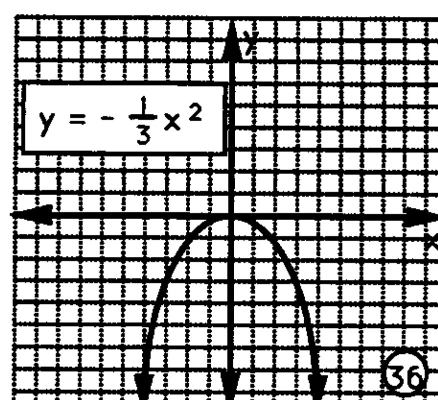

$y = -|3x|$
y
x
31

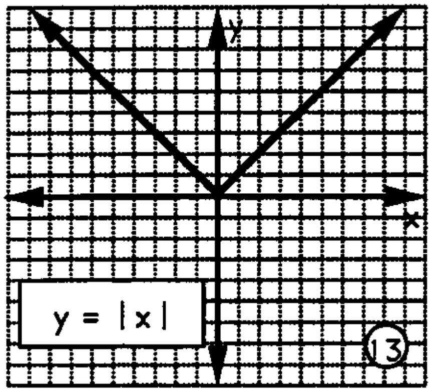

$y = x^2$
y
x
32

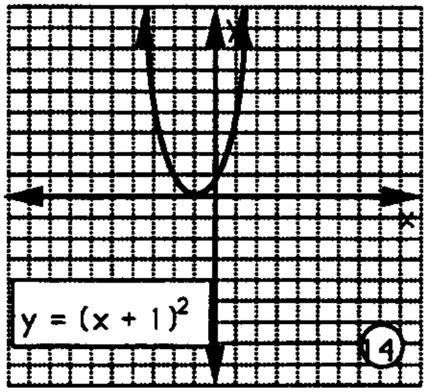

$y = \frac{1}{4}x^2$
y
x
33

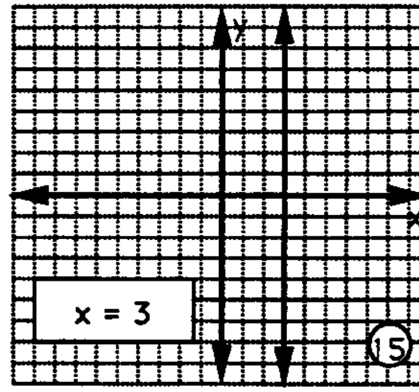

$y = -|\frac{1}{3}x|$
y
x
34

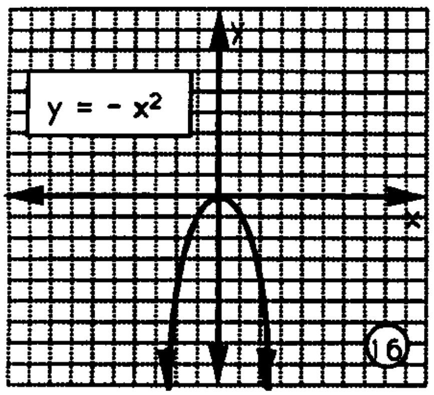

$y = -|x|$
y
x
35

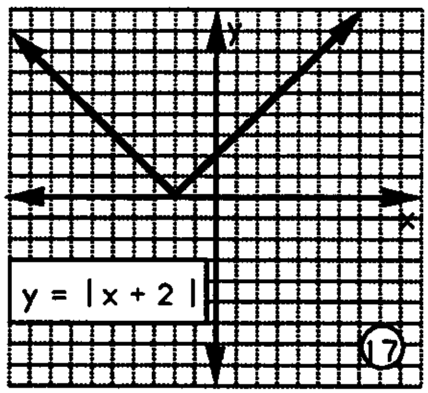

$y = -\frac{1}{3}x^2$
y
x
36

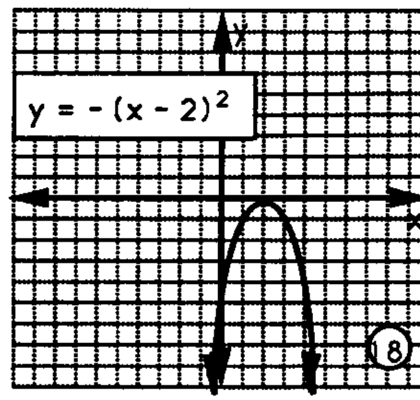

y
x
y = |x|
13

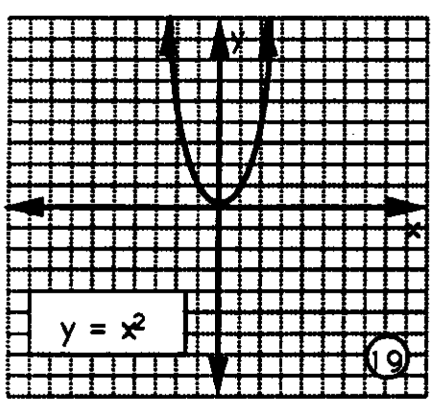

y
x
y = (x + 1)²
14

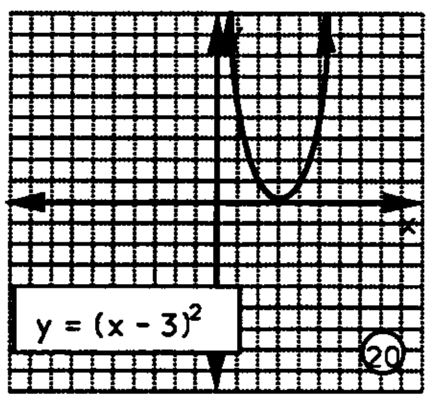

y
x
x = 3
15

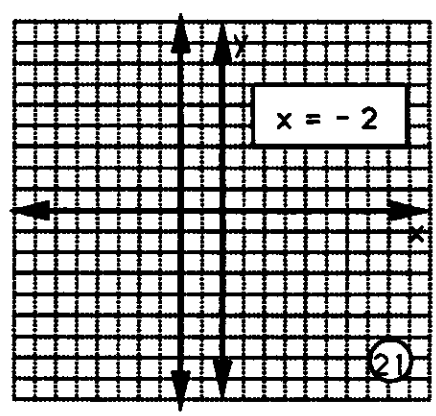

y
x
y = - x²
16

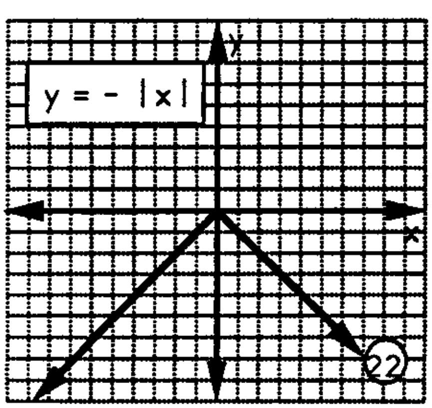

y
x
y = |x + 2|
17

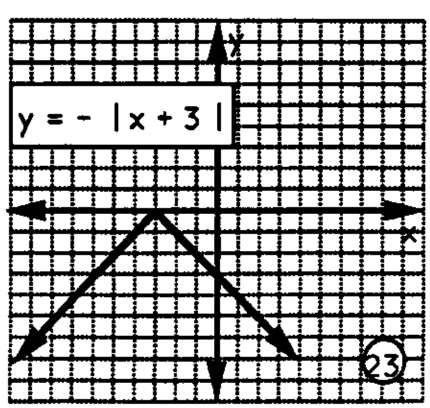

y
x
y = - (x - 2)²
18

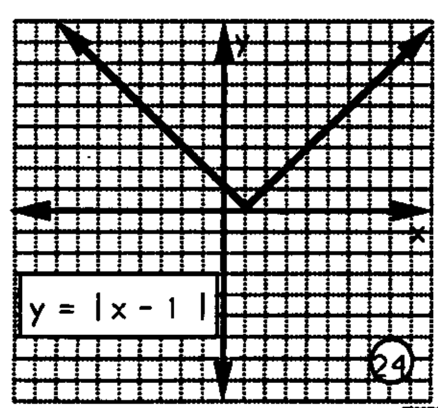

y
x
y = x²
19


x
y = (x - 3)²
20


y
x
x = - 2
21

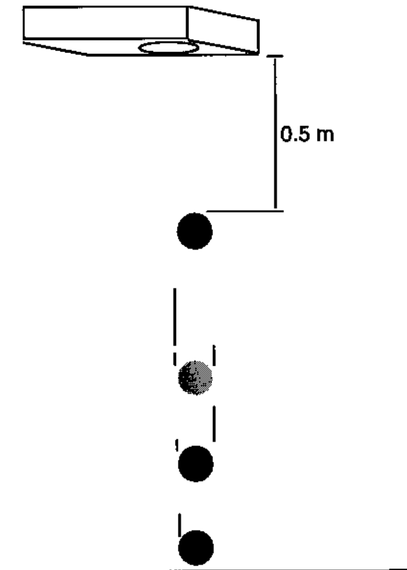

y
x
y = - |x|
22


y
x
y = - |x + 3|
23


y
x
y = |x - 1|
24

Activity 1: Collect the Data

1. Read the directions below.
   Predict the graph of the distance of the ball from the ground versus time.



2. Using a motion detector, a data collection device, and an appropriate program, do the following.

   - ❖ Hold the motion detector at least 0.5 meters above the ball.
   - ❖ Drop the ball and let it bounce under the motion detector.
   - ❖ Collect distance data for about 5 seconds.
   - ❖ Collect data for a least 5 good bounces.
   - ❖ Repeat if necessary.

3. Sketch the resulting graph:

Activity 2: A Bounce

1. Choose the first complete bounce on the graph. What kind of function would be an appropriate model for this data? Why?

2. Find a model. Write the function.

3. Sketch the data and the model.

4. List the transformations you used to find the model.

5. Choose another complete bounce and find a model for that data.

Activity 3: Bounce Height versus Bounce Number

1. Use the trace feature to find the maximum height for each full bounce. Do not fill in the height for bounce number 0.

Bounce Number	Maximum Height of Bounce
0	
1	
2	
3	
4	
5	
6	

2. Make a scatter plot of (bounce number, maximum height) in an appropriate viewing window. Sketch it. (Do not lose the original data!)

3. What kind of function would be an appropriate model? Why?

4. Find an appropriate model. Write the function. Sketch it above.

Activity 4: Bounce Height versus Drop Height

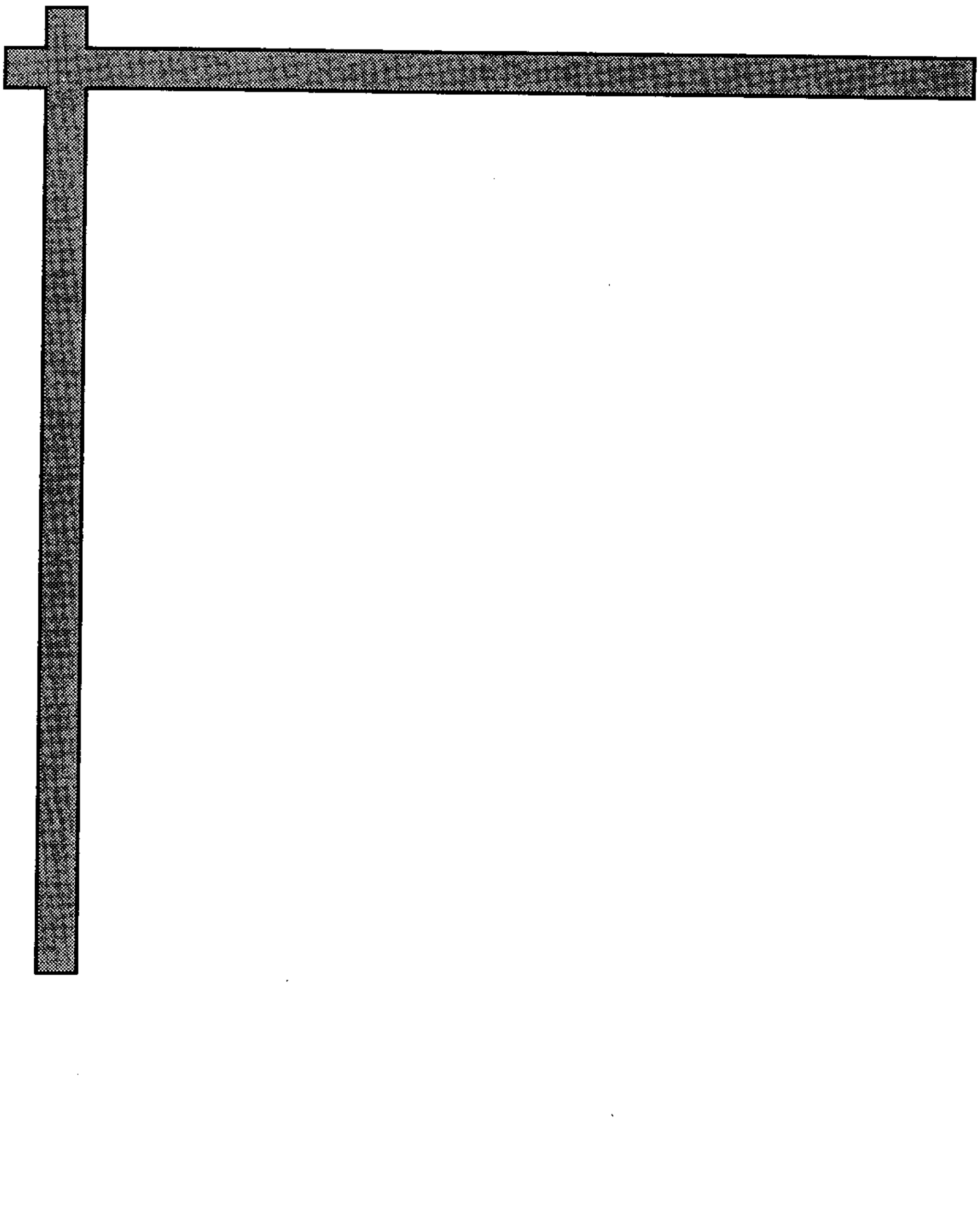
1. Using the data from Activity 3, fill in the table.

Drop Height	Bounce Height

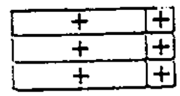
2. Make a scatter plot of (drop height, bounce height) in an appropriate viewing window. Sketch it. (Do not lose the original data!)

3. What kind of function would be an appropriate model? Why?

4. Find an appropriate model. Write the function. Sketch it above.



The Distributive Property

Problem	Model	Solution
1. $3(x+1)$		
2. $2(2x+3)$		
3. $4(x+2)$		
4. $2(x-3)$		
5. $3(2x-1)$		
6. $-2(x+3)$		
7. $-3(x+2)$		
8. $-2(x^2+1)$		

9. $-1(x + 2)$		
10. $2(x^2 + 3x)$		
11. $-2(x^2 + 3x)$		
12. $2(x + 1) + 3(x - 3)$		

Use algebra tiles to explain why $4(x - 1) = 4x + 1$.

Multiplying With Algebra Tiles

Multiply by modeling each problem.

1. $(x + 3)(x + 2) =$ ______________

2. $(x + 1)(x - 2) =$ ______________

3. $(x + 1)(x + 1) =$ ______________

4. $(x + 1)(x - 1) =$ ______________

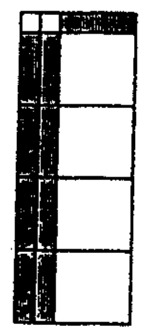
Complete the information for each model.

5. The area is ____________________
and the dimensions are _________________ and _________________.
The perimeter is _____________________.

6. The area is ____________________
and the dimensions are _________________ and _________________.
The perimeter is _____________________.

7. The area is ____________________
and the dimensions are _________________ and _________________.
The perimeter is _____________________.

8. The area is ____________________
and the dimensions are _________________ and _________________.
The perimeter is _____________________.



Review

Add or subtract as indicated.

9. $(x^2 + 5x + 4) + (3x^2 - 7x - 12)$
10. $(x^2 + 5x + 4) - (3x^2 - 7x - 12)$
11. $3xy(2x^2y^3 + 3x^3y^2 - 8xy + 2)$

Solve and graph on a number line.

12. $2x + 4 < 12$ or $3x - 4 > 11$
13. $8 - 5k > 3$ and $2k + 3 > -11$

Write the equation of the line that:

14. Passes through the points (-2,3) and (4, -7)
15. Is vertical and passes through the point (-2,4).

16. Find the value of x such that (x, 10) is a solution of $y = 3x + 4$.

Young man, there's no need to feel down
I said young man, pick yourself off the ground,
I said young man, cause you're doing new work
There's no need to feel unhappy.

Young man, there's a trick you can do.
I said young man, it can work for you too.
You can try it, and I'm sure you will find
It will work each and every time.

It's fun to do it with
**   F O I L**
It's fun to do it with
**   F O I L**
It gets every term from the first to the last
You can get it correct real fast

It's fun to do it with
**   F O I L**
It's fun to do it with
**   F O I L**
It gets every term from the first to the last
You can get it correct real fast.

Young man, are you listening to me?
I said, young man, you might just make a C.
I said, young man, you can make better grades,
But you've got to know this one thing—

No man does it all by himself.
I said, young man, put your pride on the shelf,
And just do it in four easy steps.
I said first, outer, inner and then last.

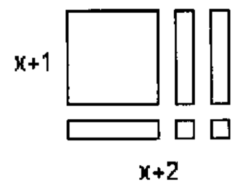
It's fun to do it with
F O I L
It's fun to do it with
F O I L
It gets every term from the first to the last
You can get it correct real fast

It's fun to do it with
F O I L
It's fun to do it with
F O I L
It's fun to do it with
F O I L
It gets every term from the first to the last
You can get it correct real fast.

Algebra Tiles - Factoring

Factor the following trinomials using algebra tiles. For each trinomial, show the arrangement of algebra tiles you used, and the binomial factors you found. For example:

Factor $x^2 + 3x + 2$



$(x+1)(x+2)$

Factor these trinomials:

$x^2 + 4x + 3$

$x^2 + 5x + 4$

$x^2 + 5x + 6$

$x^2 + 6x + 8$

$x^2 + 7x + 10$

$x^2 + 6x + 9$

$x^2 + 11x + 12$

$x^2 + 11x + 18$

$x^2 + 11x + 24$

$x^2 + 15x + 36$

ENGAGE

Students are placed in groups of three and given the following problem to solve: Create as many different rectangles as possible with an area of __________. Each group will be given a rectangle with a different area. Suggested areas are: 12, 20, 24, 30, 36, and 48. Each group will use a sheet of centimeter grid paper to create a visual representation of their rectangles. Each group will report out to the class the method they used to find all of the possible rectangles with the specified area.

EXPLORE

Students will remain in the same groups as assigned for the **ENGAGE** activity. Each group will be assigned one of the three methods to **EXPLORE** ~ The Magic Box, Factoring Using Algebra Tiles, or Factoring Using the Graphing Calculator. Details for each of the methods follow. It is especially important that the teacher circulate through the groups to offer focusing questions and to answer questions that the groups themselves cannot.

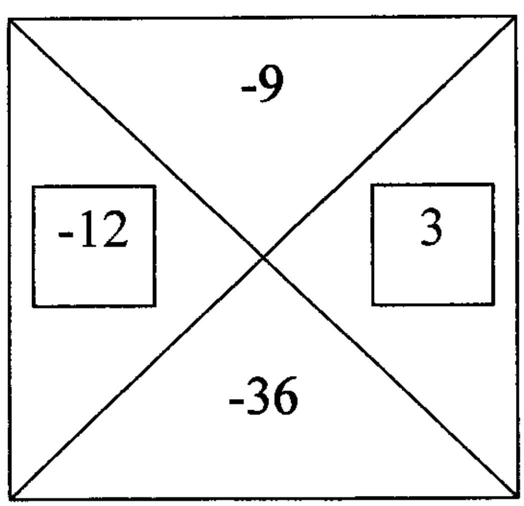
The Magic Box

Each student in the group receives a Magic Box activity sheet. The instructions are to find the pattern in the Magic Box by studying the first four examples. They must then complete the other Magic Boxes AND write a rule which explains how the Magic Box works.

The rule for the Magic Box is: Add the two numbers on the sides to get the number at the top, and multiply those same two numbers to get the number on the bottom..

After students understand how the Magic Box works, relate the Magic Box to factoring trinomials by taking a trinomial, placing the value of "b" in the top of the box and the value of "c" in the bottom of the box. Students determine the correct values for the numbers in the sides of the box. Using those numbers, write the factors of the trinomial as shown in the example below. (NOTE: The Magic Box only works for trinomials in which the value of "a" is 1.)

EXAMPLE: Find the factors of $x^2 - 9x - 36$.



Therefore the factors are $(x-12)(x+3)$.

Factoring Using Algebra Tiles

Assuming your students have had practice using algebra tiles, they should be ready to use them to factor polynomials. Through effective questioning, make sure that all students in these groups remember the value of each of the different tiles, the significance of the different colors, and what a zero pair is. As you get these groups started, be sure to tell them that this method of factoring is very similar to the ENGAGE activity and is, in fact, based upon an area model therefore their polynomials must be rectangles before they can factor them. Be sure to specify to the students that the factors will be of the form $(x \pm \textit{some number})(x \pm \textit{some number})$. They will determine the sign and value of *some number* from their model. These groups will likely require the most attention.

Factoring Using the Graphing Calculator

Students will use the function related to the polynomial to find the factors of the polynomial. For example, given the polynomial $x^2 + x - 12$, find its factors. Students should graph the function $f(x) = x^2 + x - 12$ on Y_1 on their calculators. The x-intercepts of the function can be identified either from the graph, the table, or using the trace feature. Once the intercepts are identified, it is simply a matter of finding their additive inverses and filling those numbers in for *some number* in the given form for factors ($(x \pm \textit{some number})(x \pm \textit{some number})$).

Near the end of the class period, the teacher should debrief each group to make sure all students in the group understand their method and can use it. This can be done easily by asking the group to explain one of their problems and directing questions about the method to each of the students in the group. Do NOT move to the **EXPLAIN** portion of the lesson until this debriefing has occurred.

EXPLAIN

Students are assigned to new groups of six members each. Each new group should consist of two members from each of the factoring methods groups ~ two from The Magic Box groups, two from Factoring Using Algebra Tiles groups, and two from Factoring Using the Graphing Calculator groups. Each pair is an "expert" at their method. It is their responsibility to **EXPLAIN** (teach) their method to the other group members.

ELABORATE

Students remain in their **EXPLAIN** groups and work a group of problems together. Each problem should be worked by more than one method. When all students have completed the problems, have pairs from each group explain the method they chose to use, why they chose that method, and give their answer. Allow time for discussion if others chose a different method. It is most important that students explain why and how they used a particular method. This will be the teacher's opportunity to clear up any misconceptions. This will also be the time for the teacher to stress the importance of standard form and to make sure that everyone knows that the Magic Box can only be used when the coefficient of x^2 is 1.

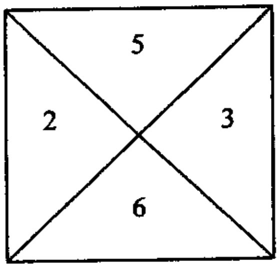
EVALUATE

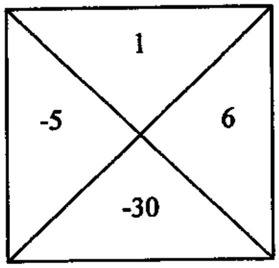
Students will complete the **EVALUATE** activity sheet independently. This assignment should be a graded assignment.

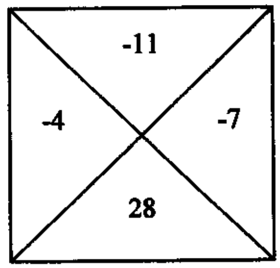
EXPLORE: THE MAGIC BOX ~ ACTIVITY SHEET

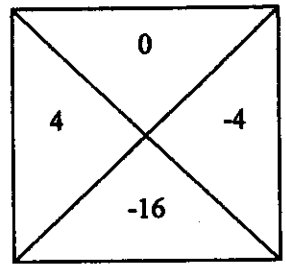
PART 1

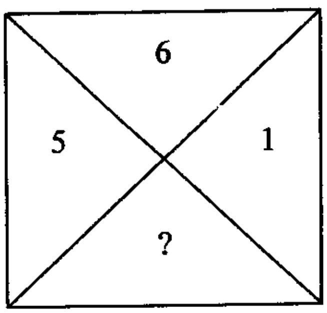
Study the Magic Boxes below and determine how the numbers are related. Once you have determined what the pattern of the Magic Boxes is, complete the other Magic Boxes AND write a sentence which explains how you were able to determine the missing numbers.

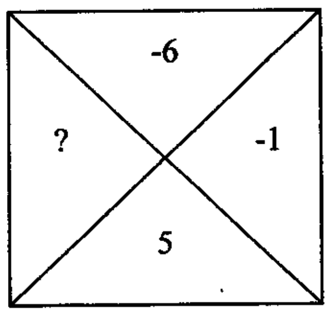


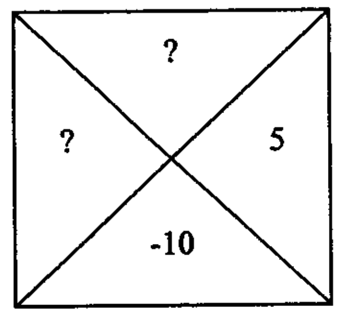


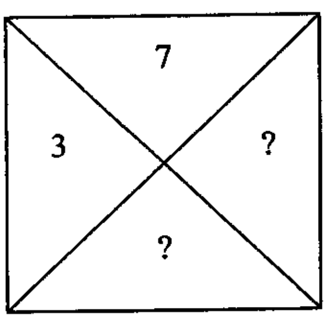


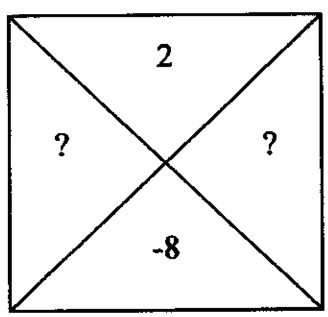


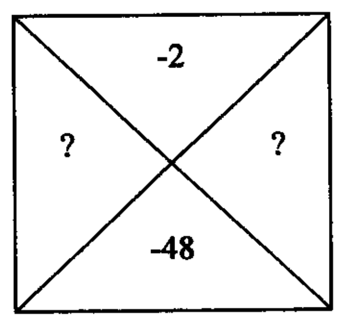












Write a sentence or two which explains how you were able to determine the missing numbers.

EXPLORE: THE MAGIC BOX ~ ACTIVITY SHEET
PART 2

Write the coefficient of the middle term in the upper part of the magic box and write the last term in the lower part of the magic box to factor the given polynomials. Write the polynomial in factored form.

$t^2+8t+12$

factors

(　　　)(　　　)

$w^2+24w+144$

factors

(　　　)(　　　)

$m^2-7m+12$

factors

(　　　)(　　　)

$n^2+3n-18$

factors

(　　　)(　　　)

$v^2-18v+80$

factors

(　　　)(　　　)

p^2-p-56

factors

(　　　)(　　　)

n^2+5n+4

factors

(　　　)(　　　)

EXPLORE: USING ALGEBRA TILES TO FACTOR ~ ACTIVITY SHEET

Use algebra tiles to factor each polynomial. Draw a sketch of the final rectangle from which you get your factors. Write the polynomial in factored form.

POLYNOMIAL	SKETCH OF RECTANGLE	FACTORED FORM
$x^2 - x - 12$		
$x^2 - 2x + 1$		
$x^2 - 1$		
$x^2 - 6x + 8$		
$x^2 + 3x + 2$		
$x^2 + 6x + 5$		

EXPLORE ~ FACTORING USING THE GRAPHING CALCULATOR ~ ACTIVITY SHEET

Use your graphing calculator to factor each of the given polynomials. Write the polynomial in factored form.

POLYNOMIAL	RELATED FUNCTION	X-INTERCEPTS	FACTORED FORM
x^2+3x-4			
$x^2+10x+9$			
$x^2-11x+30$			
x^2-x-12			
$x^2+7x+12$			
$x^2-9x+18$			

ELABORATE ~ ACTIVITY SHEET

Factor the given polynomials by any method you choose. Tell which method you used and why you chose it. Write the polynomial in factored form.

POLYNOMIAL	METHOD USED	EXPLANATION	FACTORED FORM
$x^2 - 11x + 24$			
$x^2 + 14x - 15$			
$x^2 + 7x + 10$			
$x^2 - 24x + 108$			
$x^2 + 26x + 169$			

EVALUATE ~ ACTIVITY SHEET

Factor the given polynomial by any method you choose. Tell which method you used and why you chose it. Write the polynomial in factored form.

POLYNOMIAL	METHOD USED	EXPLANATION	FACTORED FORM
$x^2+6x-91$			

CHALLENGE

Factor the given polynomial either by using algebra tiles or your graphing calculator. Tell which method you used and why you chose it. Write the polynomial in factored form.

POLYNOMIAL	METHOD USED	EXPLANATION	FACTORED FORM
$2x^2-9x+4$			

ALGEBRA TIC-TAC-TIMES Game Board

from *The Mathematics Teacher*

$x^2 - 7x + 12$	$x^2 - 3x + 2$	$x^2 - 16$	$x^2 + 8x + 16$	$x^2 - x$
$x^2 + 5x + 4$	$x^2 - 4x$	$x^2 + 2x - 3$	$x^2 + x$	$x^2 - 1$
$x^2 - 8x + 16$	$x^2 - 5x + 6$	$x^2 - 4x + 4$	$x^2 + 7x + 12$	$x^2 - 2x - 8$
$x^2 - 4$	$x^2 + 2x$	$x^2 - 6x + 9$	$x^2 - 9$	$x^2 + 3x - 4$
$x^2 - 2x + 1$	$x^2 - 2x - 3$	$x^2 - 2x$	x^2	$x^2 + 5x + 6$
$x^2 - 6x + 8$	$x^2 + 4x + 4$	$x^2 + 2x - 8$	$x^2 + 3x$	$x^2 - 4x + 3$
$x^2 + 6x + 9$	$x^2 + x - 2$	$x^2 + 4x + 3$	$x^2 - x - 2$	$x^2 - 3x$
$x^2 - 3x - 4$	$x^2 + x - 12$	$x^2 - x - 6$	$x^2 + 4x$	$x^2 + 6x + 8$
$x^2 + 3x + 2$	$x^2 + 2x + 1$	$x^2 - 5x + 4$	$x^2 - x - 12$	$x^2 + x - 6$

Game Board

$x - 4$	$x - 3$	$x - 2$	$x - 1$	x	$x + 1$	$x + 2$	$x + 3$	$x + 4$

Factor Board

ALGEBRA TIC-TAC-TIMES

by Richard J. Crouse and Marilyn J. Sweeney
from *The Mathematics Teacher*

"Algebra tic-tac-times" combines mathematical skills with a competitive strategy. It is a highly motivational skill-review exercise that involves the problem-solving strategy of working backward.

Directions for Play

Objective
The object of the game is similar to that of tic-tac-toe; the winner is the first of two players to place four tokens in a row, either vertically, horizontally, or diagonally.

Materials
The materials necessary to play algebra tic-tac-times include a factor board and a game board (fig. 1) and forty translucent tokens of two different colors. One token of each color is used as the factor marker, and the remainder are used as game tokens. The game board should be laminated so that it can be saved from year to year. The tokens should be stored in a bag so that they don't get lost.

Method of play
Player 1 begins the game by placing a factor marker and one of player 2's factor markers on any factors on the factor board. The product of these factors determines the placement of player 1's game token. In figure 1, player 1 placed a factor marker on x + 1 and player 2's marker on x -1. Player 1 then placed a game token on x^2 -1 because (x -1)(x + 1) = x^2 -1.

$x^2-7x+12$	x^2-3x+2	x^2-16	$x^2+8x+16$	x^2-x
x^2+5x+4	x^2-4x	x^2+2x-3	x^2+x	x^2-
$x^2-8x+16$	x^2-5x+6	x^2-4x+4	$x^2+7x+12$	x^2-2x-8
x^2-4	x^2+2x	x^2-6x+9	x^2-9	x^2+3x-4
x^2-2x+1	x^2-2x-3	x^2-2x	x^2	x^2+5x+6
x^2-6x+8	x^2+4x+4	x^2+2x-8	x^2+3x	x^2-4x+3
x^2+6x+9	x^2+x-2	x^2+4x+3	x^2-x-2	x^2-3x
x^2-3x-4	x^2+x-12	x^2-x-6	x^2+4x	x^2+6x+8
x^2+3x+2	x^2+2x+1	x^2-5x+4	x^2-x-12	x^2+x-6

Game Board

$x-4$	$x-3$	$x-2$	1	x	1	$x+2$	$x+3$	$x+4$

Factor Board

Player 1's game token Player 2's game token

Fig. 1. Player 1's first move

Note: factor markers can be placed on the same factor, resulting in squared factors.

Player 2 can move only player 2's factor marker (player l's marker remains in place) to another factor on the factor board, as shown in figure 2. In this example, player 2 could move a factor marker to x. The product of these new factors determines the placement of player 2's game token. In this example, player 2 would place a game token on the product of x + 1 (player l's marker) and x (player 2's marker), or x^2 +

x, on the game board.

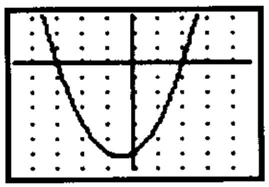
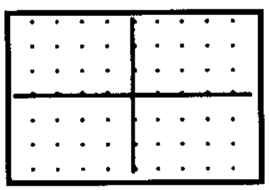
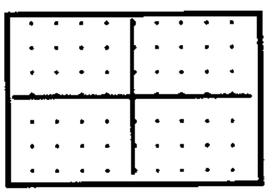
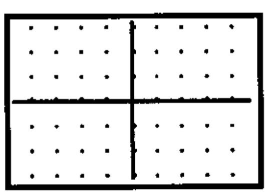
Players must use a strategy of working backward to determine which products com- bined with the available factors will win the game. These same problem-solving strate- gies become a part of the defensive play of the game when a player wishes to block an opponent.

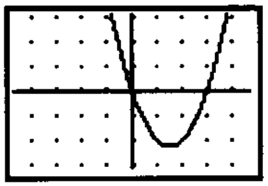
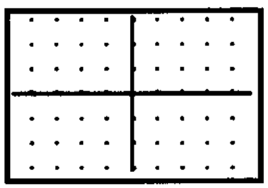
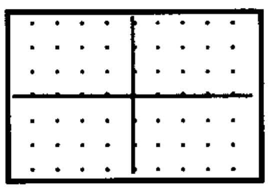
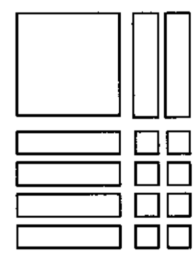
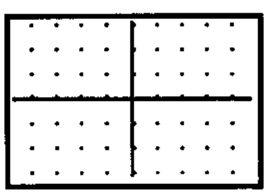
Penalties

A player is penalized when a product that has already been covered is used or when an incorrect response to the factors is given. A move is considered to be completed when a player's hand is removed from the factor marker in the event of a duplicate product or from the game token in the event of an incorrect response. In the event of a penalty, the opposing player has the opportunity to move both factor markers, as in the begin- ning of the game.

Click here for a copy of the Game Board

8. Complete the table.

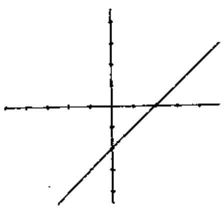
Graph y_4	y_1	y_2	y_3 (factored form)	Roots	y_4 (polynomial form)	Solution(s) to $y_4 = 0$
	$x+3$	$x-2$	$(x+3)(x-2)$	- 3 and 2	x^2+x-6 	$x=-3, 2$
	$x+1$	$x+2$				
			$(x-4)(x-1)$			
				-3 and -4		

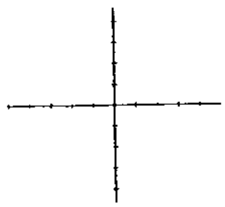
Graph y_4	y_1	y_2	y_3 (factored form)	Roots	y_4 (polynomial form)	Solution(s) to $y_4 = 0$
						
					$y = x^2 - x - 12$	
						
						$x = -2, 3$

Graphing One Step At A Time
Composition of Functions

Given the graph of $g(x)$ and given $f(x)$, graph $f(g(x))$.

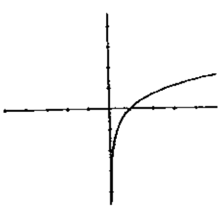
1. $g(x)$; $f(x) = x^2$ $f(g(x))$

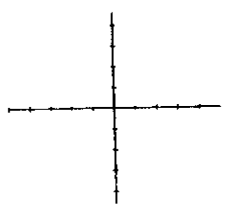




Notice that $g(x)$ represents y-values on the graph thus $f(g(x))$ means square the y-values of the graph of $g(x)$ at corresponding x-values. Therefore where $g(x)$ is negative it becomes positive (stretching or compressing as appropriate), where $g(x)$ is positive it remains positive (stretching or compressing as appropriate) and where $g(x)$ is zero it remains zero. Note the graph is $y = (x-2)^2$, but notice how it is not approached as a shift right instead it is a shift down and then the effect of the square, a nice connection.

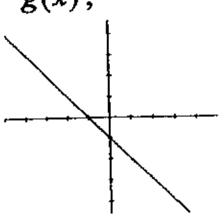
2. $g(x)$; $f(x) = x^2$ $f(g(x))$

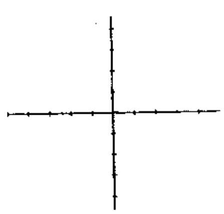




Very similar to the first example

3. $g(x)$; $f(x) = \sqrt{x}$ $f(g(x))$





Since $f(x) = \sqrt{x}$ domain includes only positive real numbers and zero ($x \geq 0$) wherever $g(x)$ is negative there will not be any values. $g(x)$ is positive or zero for $x \leq -1$ so $f(g(x))$ will only exist on that domain.

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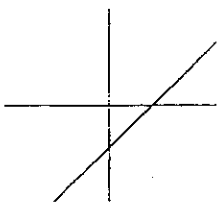
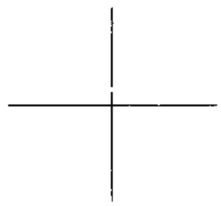
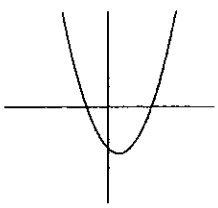
For each exercise you are given the graph of a function, $g(x)$. Given $f(x)$ graph $f(g(x))$.

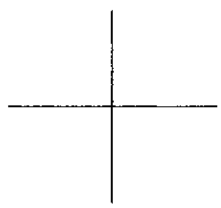
3. $g(x)$; $f(x) = x^2$ $f(g(x))$



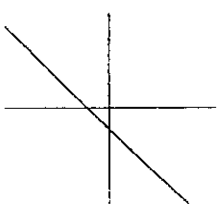


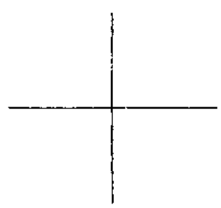
4. $g(x)$; $f(x) = \sqrt{x}$ $f(g(x))$



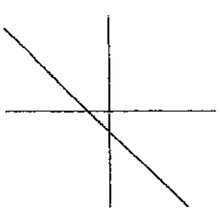


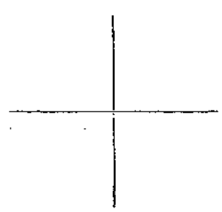
5. $g(x)$; $f(x) = \ln(x)$ $f(g(x))$





6. $g(x)$; $f(x) = 2^x$ $f(g(x))$





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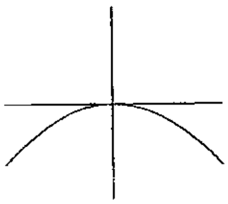
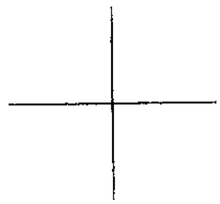
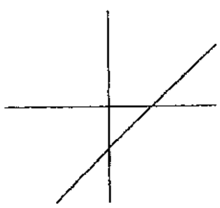
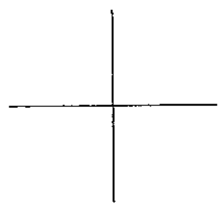
7. $g(x)$; $f(x)=\frac{1}{x}$ $f(g(x))$



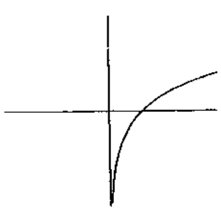


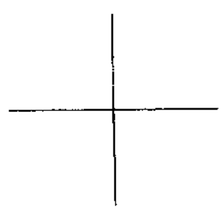
8. $g(x)$; $f(x)=\frac{1}{x}$ $f(g(x))$



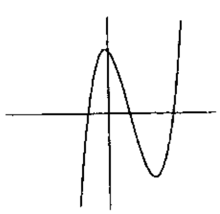


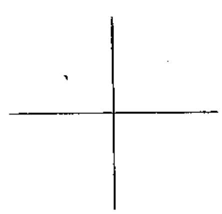
9. $g(x)$; $f(x)=x^2+1$ $f(g(x))$





10. $g(x)$; $f(x)=|x|$ $f(g(x))$





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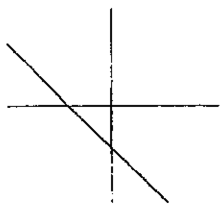
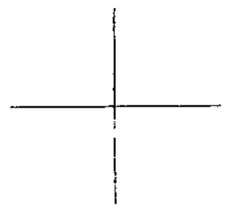
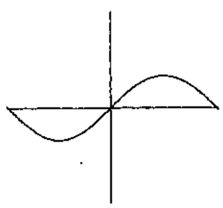
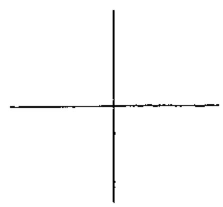
11. $g(x)$; $f(x) = \ln x$ $f(g(x))$





12. $g(x)$; $f(x) = (x+1)^2$ $f(g(x))$



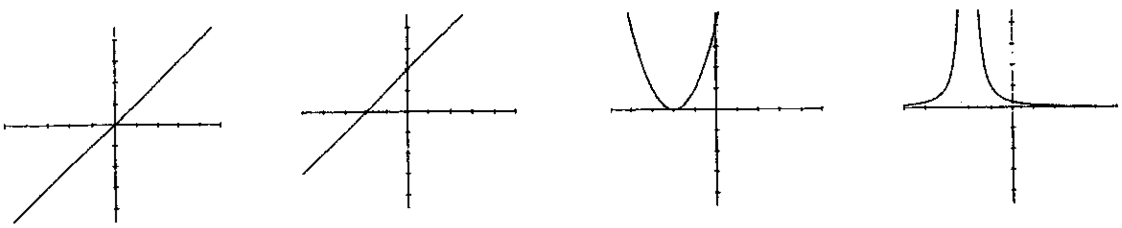


The idea of graphing compositions one step at a time can be used to graph functions. For instance, $\dfrac{1}{(x+2)^2}$ is $f_1(x) = x$, $f_2(x) = x+2$, $f_3(x) = x^2$ and $f_4(x) = \dfrac{1}{x}$ where

$$f_4(f_3(f_2(f_1(x)))) = \frac{1}{(x+2)^2}$$

Considering the composition step by step, the graph of $f(x)=\dfrac{1}{(x+2)^2}$ can be determined as follows;

1^{st} $y=x$ 2^{nd} add 2 3^{rd} square 4^{th} invert



Determine the order of composition and graph the following one step at a time using the process above.

13. $j(x)=\sqrt{x+2}$

14. $g(x)=\left(\left|x^2-1\right|-1\right)^3$

15. $h(x)=\sqrt{2-x^2}$

16. $f(x)=-\dfrac{1}{1+x^2}$

17. $f(x)=\dfrac{1}{(1+\cos x)}$

18. $g(x)=\ln(x^2+1)$

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Composition of Functions

Composition of Functions Warm Up

1. a) If $f(x)$ and $g(x)$ are polynomial functions, use the two tables of values below to complete the table of values for $f(g(x))$.

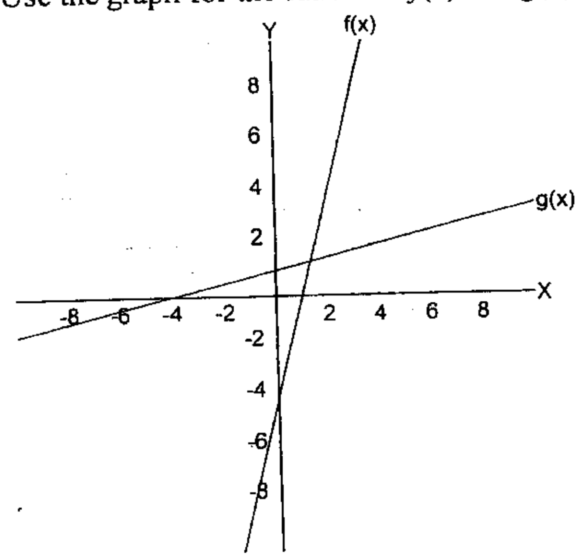
x	$f(x)$
0	3
1	4
2	5
3	6
4	7

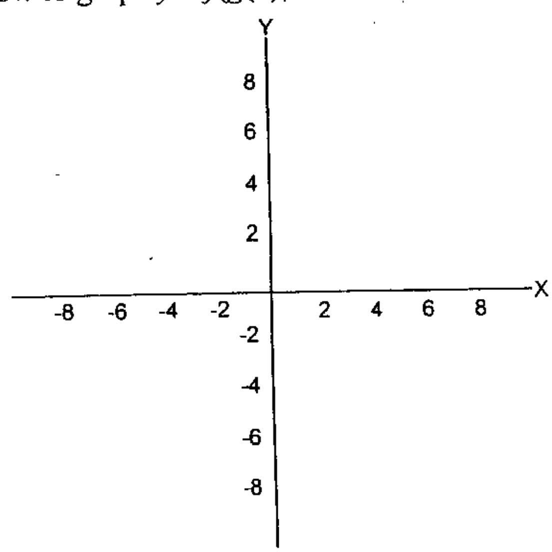
x	$g(x)$
-2	4
-1	1
0	0
1	1
2	4

x	$f(g(x))$
-2	
-1	
0	
1	
2	

b) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

2. a) Use the graph for the functions $f(x)$ and $g(x)$ below to graph $y = f(g(x))$.





b) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

Composition of Functions Worksheet

1. a) If $f(x)$ and $g(x)$ are linear functions, use the two tables of values below to complete the table of values for $f(g(x))$.

x	$f(x)$
-2	-6
-1	-3
0	0
1	3
2	6

x	$g(x)$
-6	-2
-3	-1
0	0
3	1
6	2

x	$f(g(x))$
-6	
-3	
0	
3	
6	

b) Write mathematical models for $f(x)$ and $g(x)$.

c) Find f^{-1} and g^{-1} .

d) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

2. a) If f(x) and g(x) are linear functions, use the two tables of values below to complete the table of values for $f(g(x))$.

x	$f(x)$
-2	-8
-1	-6
0	-4
1	-2
2	0

x	$g(x)$
-8	-2
-6	-1
-4	0
-2	1
0	2

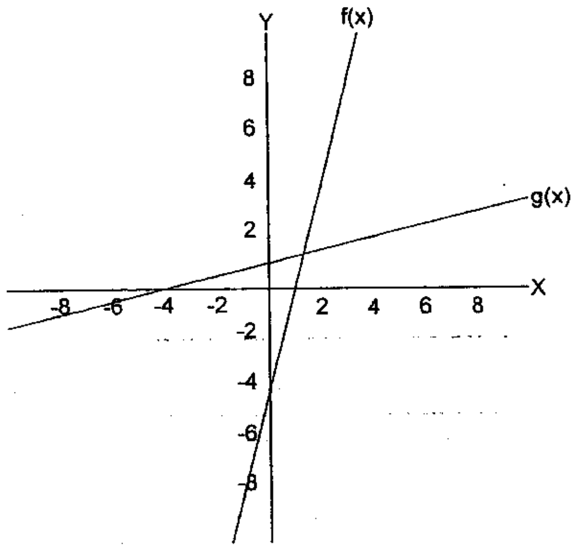
x	$f(g(x))$
-8	
-6	
-4	
-2	
0	

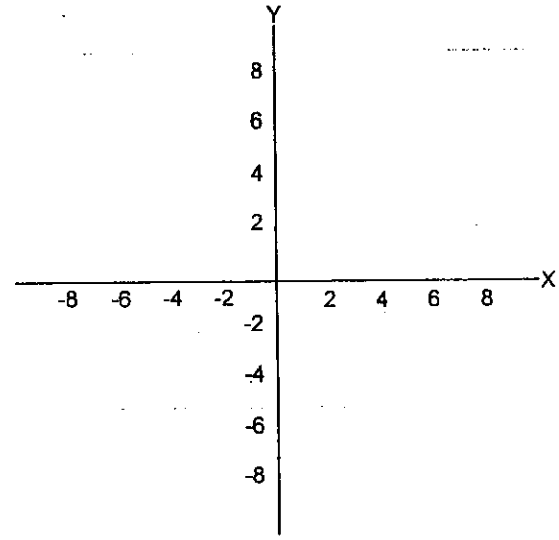
b) Write mathematical models for $f(x)$ and $g(x)$.

c) Find f^{-1} and g^{-1} .

d) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

3. a) Use the graph for the functions $f(x)$ and $g(x)$ below to graph $y = f(g(x))$.



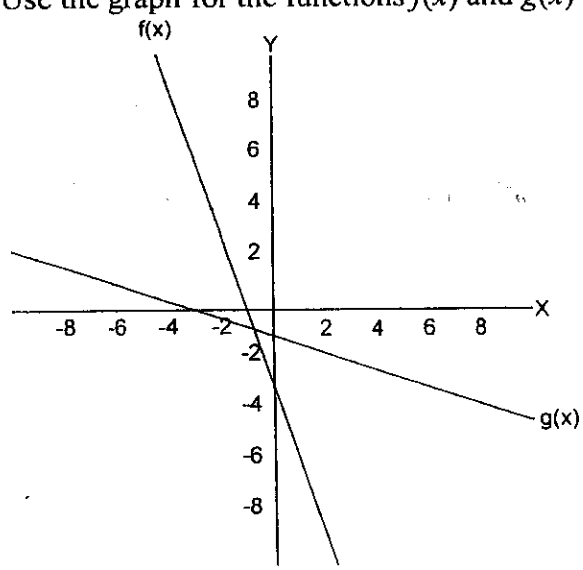


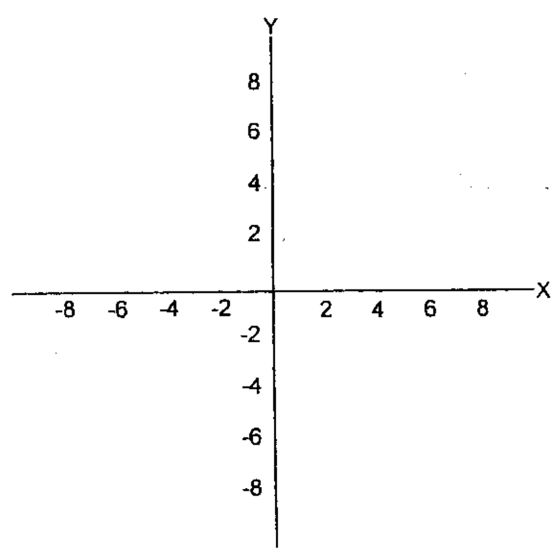
b) Write mathematical models for $f(x)$ and $g(x)$.

c) Find f^{-1} and g^{-1} .

d) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

4. a) Use the graph for the functions $f(x)$ and $g(x)$ below to graph $y = f(g(x))$.





b) Write mathematical models for $f(x)$ and $g(x)$.

c) Find f^{-1} and g^{-1} .

d) Let $h(x) = f(g(x))$. Write a mathematical model for $h(x)$.

5. What do you observe about the composition of inverse functions? Why do you think this happens?

6. a) If the function g is the inverse of the function f predict $f(g(x))$ and $g(f(x))$.

   b) Check your prediction given $g(x) = 2x$ and $f(x) = \frac{1}{2}x$.

7. Given $f(x) = \sqrt[3]{x + 1}$ and $g(x) = x^3 + 1$ show that $g(x)$ and $f(x)$ are not inverse functions.

8. Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$, determine if $g(x)$ and $f(x)$ inverse functions? Show the analysis that leads to your conclusion.

Composition of Functions

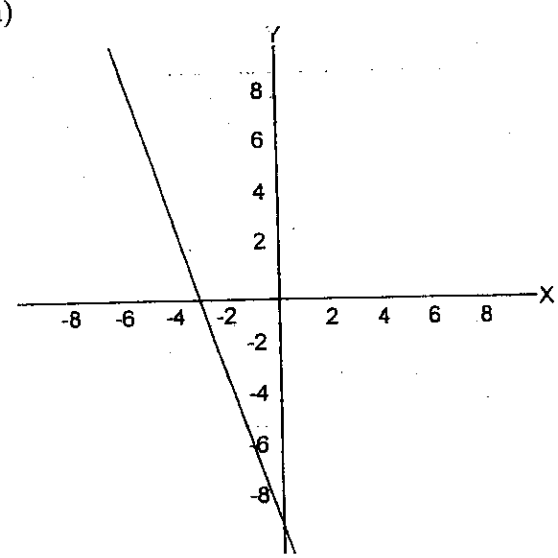
Composition of Functions Warm-up

1 a)

x	$f(g(x))$
-2	7
-1	4
0	3
1	4
2	7

b) $h(x) = x^2 + 3$

2 a)



b) $h(x) = -3x - 9$

Composition of Functions Worksheet

1. a)

x	$f(g(x))$
-6	-6
-3	-3
0	0
3	3
6	6

b) $f(x) = 3x$ $\quad g(x) = -\frac{1}{3}x$

c) $f^{-1}(x) = -\frac{1}{3}x$ $\quad g^{-1}(x = 3x)$

d) $h(x) = x$

2. a)

x	$f(g(x))$
-8	-8
-6	-6
-4	-4
-2	-2
0	0

b) $f(x) = 2x - 4$ $\quad g(x) = x + 2$

c) $f^{-1}(x) = x + 2$ $\quad g^{-1}(x) = 2x - 4$

d) $h(x) = x$

Activity 1: Explore Transformations

I. Exploration I

1. Enter: $y_2 = y_1 + 1$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graph of y_1 and y_2.

a. $y_1 = |x|$ b. $y_1 = \ln x$ c. $y_1 = \cos x$ d. $y_1 = -x + 2$

2. Enter: $y_2 = y_1 - 2$.

a. $y_1 = 0.5x$ b. $y_1 = \sin x$ c. $y_1 = 2^x$ d. $y_1 = -|x|$

3. Generalize: What happens to the graph of a function when you add a constant to the function rule?

II. Exploration II

1. Enter: $y_2 = y_1\,(x - 1)$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graphs of y_1 and y_2.

a. $y_1 = |x|$ b. $y_1 = x^2$ c. $y_1 = \frac{1}{x}$ d. $y_1 = x + 1$

2. Enter: $y_2 = y_1\,(x + 2)$.

a. $y_1 = \sqrt{9 - x^2}$ b. $y_1 = \sin x$ c. $y_1 = x^3$ d. $y_1 = \sqrt{x}$

3. Generalize: What happens to the graph of a function when you replace x with $x - a$?

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III. Exploration III

1. Enter: $y_2 = 3y_1$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graphs of y_1 and y_2.

a. $y_1 = e^x$ b. $y_1 = x^3$ c. $y_1 = \cos x$ d. $y_1 = x$

2. Enter: $y_2 = \frac{1}{3} y_1$.

a. $y_1 = \sqrt{x}$ b. $y_1 = \sin x$ c. $y_1 = e^x$ d. $y_1 = |x|$

3. Generalize: What happens to the graph of a function when you multiply the function rule by a?

IV. Exploration IV

1. Enter: $y_2 = y_1\,(2x)$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graphs of y_1 and y_2.

a. $y_1 = \cos x$ b. $y_1 = x^2$ c. $y_1 = |x|$ d. $y_1 = \sqrt{9 - x^2}$

2. Enter: $y_2 = y_1\left(\frac{1}{2}x\right)$

a. $y_1 = \sin x$ b. $y_1 = x^3$ c. $y_1 = -|x|$ d. $y_1 = \sqrt{9 - x^2}$

3. Generalize: What happens to the graph of a function when you replace x with ax?

V. Exploration V

1. Enter: $y_2 = -y_1$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graphs of y_1 and y_2.

a. $y_1 = |x|$ b. $y_1 = \ln x$ c. $y_1 = \cos x$ d. $y_1 = x^2$

2. Generalize: What happens to the graph of a function when you multiply the function by -1?

3. Enter: $y_2 = y_1(-x)$.

a. $y_1 = \sqrt{x}$ b. $y_1 = x^2$ c. $y_1 = 2^x$ d. $y_1 = x^3$

4. Generalize: What happens to the graph of a function when you replace x with $-x$?

VI. Exploration VI

1. Enter: $y_2 = |y_1|$.
   Enter the following functions, one at a time, into y_1.
   Use a friendly window.
   Sketch the graphs of y_1 and y_2.

a. $y_1 = x^2 - 2$ b. $y_1 = \ln x$ c. $y_1 = \cos x$ d. $y_1 = x$

2. Generalize: What happens when you graph the absolute value of a function?

3. Enter: $y_2 = y_1(|x|)$.

a. $y_1 = \ln x$ b. $y_1 = \sin x$ c. $y_1 = 2^x$ d. $y_1 = x^2$

4. Generalize: What happens to the graph of a function when you replace x with $|x|$?

5. How do absolute value compositions compare with the reflections in Exploration V?

Name:

Summary of Transformations

Be sure to include both a written and graphical description.

I. $y = f(x) + c$ $y = f(x) - c$

II. $y = f(x - c)$ $y = f(x + c)$

III. $y = cf(x)$ $y = \frac{1}{c} f(x)$

Name:

IV. $y = f(cx)$

$y = f(\frac{1}{c}x)$

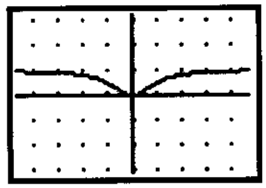
V. $y = -f(x)$

$y = f(-x)$

VI. $y = |f(x)|$

$y = f(|x|)$

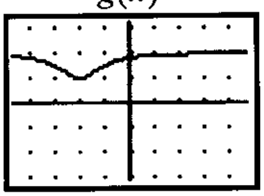
Reflect and Apply



Let $f(x) = \dfrac{x^2}{x^2+1}$

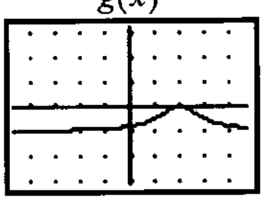
Write the function rule for $g(x)$ in terms of $f(x)$ **and** in terms of x for each of the following graphs:

$g(x)$



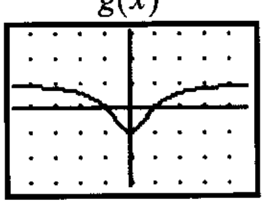
1. ____________________ ____________________

$g(x)$



2. ____________________ ____________________

$g(x)$



3. ____________________ ____________________

Transformation Practice 3

Function: $y=\sqrt{x+4}$
Window:
Domain:

Range:

a. Predict the graph of each transformation.
b. Describe the transformation in words.
c. Identify effects on the domain and range.

1. $y=f(x)-3$

a.

b.

c.

2. $y=f(x-3)$

a.

b.

c.

3. $y=2f(x)$

a.

b.

c.

4. $y=-f(x)$

a.

b.

c.

Frantic Functions

New: 08/15/08

Objective:
Students will practice making connections between formulas, graphs, tables, and verbal descriptions of parent functions.

Connections to Previous Learning:
Students should be familiar with transformations on a variety of parent functions.

Connections to AP*:
AP Calculus Topics: Analysis of Functions

Materials:
Student Activity pages are formatted so they can be printed on perforated business card stock (2″ × 3.5″). They can also be printed on cardstock or regular paper and cut into cards. The "Frantic Function" page is intended to be copied on the backs of the cards. Calculators are not allowed.

Teacher Notes:
This matching card game provides students practice in multiple representations for parent functions and their transformations. Four cards form a "set" that consists of the graph, a table of select values, a verbal description of the transformation, and an equation. The cards are marked with a letter or number in the lower right hand corner. A student answer page is available for students to record their matches in an organized fashion for ease in grading. The game can be effectively played by groups of three or four students

The table cards may be difficult to match for some of the functions. Remind students to look for trends in the table values. In some tables, a key point has been omitted in order not to give away the answers easily.

If students are not familiar with some of the types of functions, those cards can be omitted from the game and their rows marked off the answer page. As students are introduced to the new types of functions, their corresponding cards can be returned to the game set.

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

Frantic

Functions

$$f(x) = 2|x+1| - 1$$

m

$$f(x) = -\sqrt{x} - 2$$

f

$$f(x) = 2\sqrt{-x} + 1$$

p

$$f(x) = \frac{-2}{(x+1)} + 2$$

d

$$f(x) = -\frac{1}{2}x^2 - 1$$

n

$$f(x) = 3x + 2$$

c

$$f(x) = -\frac{1}{2}(x-1)^2 + 1$$

a

$$f(x) = x - 2$$

i

$$f(x) = \sqrt{x-2}$$

s

$$f(x) = x^3 - 2$$

t

$$f\left(x\right)=-2\left[x\right]+2$$

e

$$f\left(x\right)=\left|x-2\right|$$

r

$$f\left(x\right)=\frac{1}{x-2}$$

b

$$f\left(x\right)=-\left(x-2\right)^3$$

g

$$f\left(x\right)=\left[x-2\right]$$

k

$$f\left(x\right)=x^2-2$$

l

$$f\left(x\right)=-\left(x+2\right)^2-1$$

j

$$f(x)=2(x-3)+1$$

h

$$f(x)=-\frac{1}{2}(x+1)-2$$

o

$$f(x)=-\frac{1}{4}\left|x+2\right|$$

q

Absolute value; horizontal translation 1 unit left; vertical stretch of 2; vertical translation 1 unit down

B

Square root; reflection across the x-axis; vertical translation 2 units down

Q

Square root; reflection across the y-axis; vertical stretch of 2; vertical translation 1 unit up

F

Rational; horizontal translation 1 unit left; vertical stretch of 2; reflection across the x-axis; vertical translation 2 units up

A

Quadratic; vertical shrink of $\frac{1}{2}$; reflection across the x-axis; vertical translation 1 unit down

H

Linear; vertical stretch of 3; vertical translation 2 units up

S

Quadratic; horizontal translation 1 unit right; vertical shrink of $\frac{1}{2}$; reflection across the x-axis; vertical translation 1 unit up

L

Linear; vertical translation 2 units down

J

Square root; horizontal translation 2 units right

M

Cubic; vertical translation 2 units down

I

Greatest Integer; vertical stretch of 2; reflection across the x-axis; vertical translation 2 units up

G

Absolute value; horizontal translation 2 units right

K

Rational; horizontal translation 2 units right

C

Cubic; horizontal translation 2 units right; reflection across the x-axis

R

Greatest Integer; horizontal translation 2 units right

O

Quadratic; vertical translation 2 units down

T

Quadratic; horizontal translation 2 units left; reflection across the x-axis; vertical translation 1 unit down

E

Linear; horizontal translation 3 units right; vertical stretch of 2; vertical translation 1 unit up

D

Linear; horizontal translation 1 unit left; vertical shrink of $\frac{1}{2}$; reflection across the x-axis; vertical translation 2 units down

N

Absolute value; horizontal translation 2 units left; vertical shrink of $\frac{1}{4}$; reflection across the x-axis

P

X	Y_1
-3	3
-2	1
-1	-1
0	1
1	3
2	5
3	7

X= -3

2

X	Y_1
-1	ERROR
0	-2
1	-3
2	-3.414
3	-3.732
4	-4
5	-4.236

X= -1

10

X	Y_1
-5	5.4721
-4	5
-3	4.4641
-2	3.8284
-1	3
0	1
1	ERROR

X= -5

4

X	Y_1
-5	2.5
-4	2.6667
-3	3
-2	4
-1	ERROR
0	0
1	1

X= -5

7

X	Y_1
-5	-13.5
-4	-9
-3	-5.5
-2	-3
-1	-1.5
0	-1
1	-1.5

X= -5

8

X	Y_1
-5	-13
-4	-10
-3	-7
-2	-4
-1	-1
0	2
1	5

X= -5

3

X	Y_1
-5	-17
-4	-11.5
-3	-7
-2	-3.5
-1	-1
0	.5
1	1

X= -5

15

X	Y_1
-5	-7
-4	-6
-3	-5
-2	-4
-1	-3
0	-2
1	-1

X= -5

9

X	Y_1
1	ERROR
2	0
3	1
4	1.4142
5	1.7321
6	2
7	2.2361

X=1

11

X	Y_1
1	-1
2	6
3	25
4	62
5	123
6	214
7	341

X=1

20

X	Y_1
1	0
2	-2
3	-4
4	-6
5	-8
6	-10
7	-12
X=1	

19

X	Y_1
1	-1
2	ERROR
3	1
4	.5
5	.33333
6	.25
7	.2
X=1	

13

X	Y_1
1	1
2	0
3	1
4	2
5	3
6	4
7	5
X=1	

1

X	Y_1
1	1
2	0
3	-1
4	-8
5	-27
6	-64
7	-125
X=1	

16

X	Y_1
1	-1
2	0
3	1
4	2
5	3
6	4
7	5
X=1	

5

X	Y_1
1	-1
2	2
3	7
4	14
5	23
6	34
7	47
X=1	

6

X	Y_1
1	-10
2	-17
3	-26
4	-37
5	-50
6	-65
7	-82
X=1	

14

X	Y_1
1	-3
2	-1
3	1
4	3
5	5
6	7
7	9
X=1	

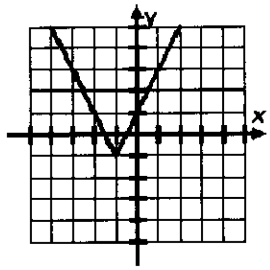
18

X	Y_1
-5	0
-4	-.5
-3	-1
-2	-1.5
-1	-2
0	-2.5
1	-3
X= -5	

12

X	Y_1
-2	0
-1	-.25
0	-.5
1	-.75
2	-1
3	-1.25
4	-1.5
X= -2	

17



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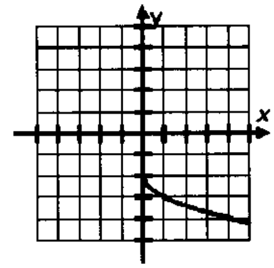
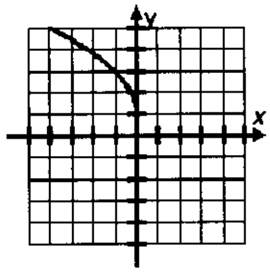


I



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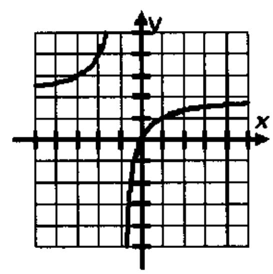

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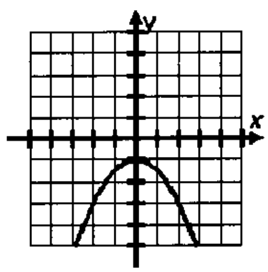

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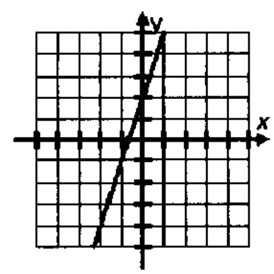

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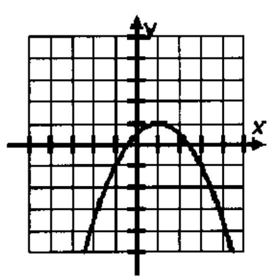

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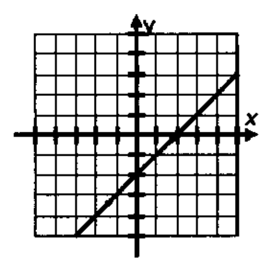

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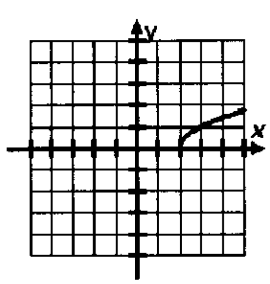

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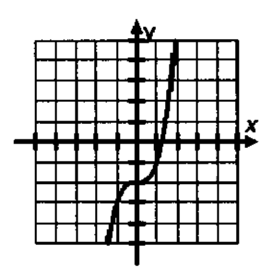

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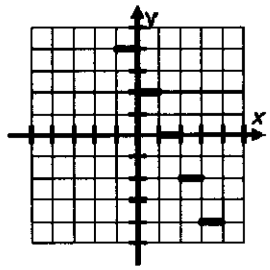

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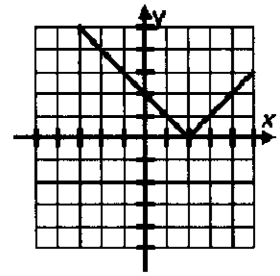

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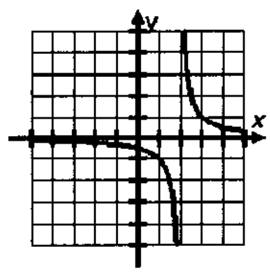

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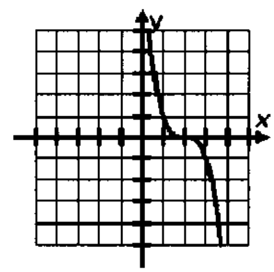

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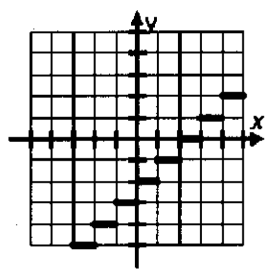

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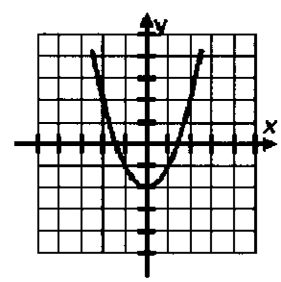

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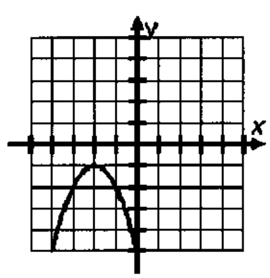

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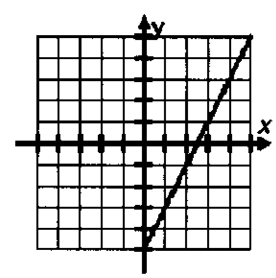

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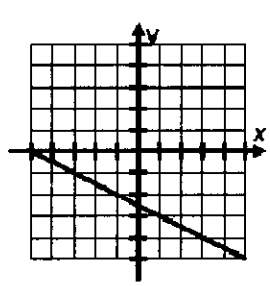

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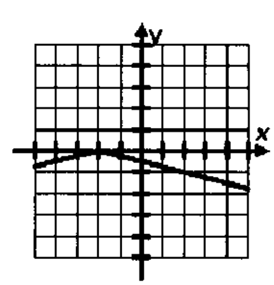


XVII

After you have matched the cards, complete the following table with the letters or numbers in the right hand corner of the cards.

Formula	Description	Table	Graph
a			
b			
c			
d			
e			
f			
g			
h			
i			
j			
k			
l			
m			
n			
o			
p			
q			
r			
s			
t			

Frantic Functions

Answers:

Each card is marked with a letter or number in the lower right corner. The following table matches the correct cards.

Formula	Description	Table	Graph
a	L	15	XV
b	C	13	XIV
c	S	3	XI
d	A	7	IV
e	G	19	X
f	Q	10	I
g	R	16	XIX
h	D	18	III
i	J	9	XX
j	E	14	XVI
k	O	5	VIII
l	T	6	XVIII
m	B	2	VII
n	H	8	XIII
o	N	12	XII
p	F	4	VI
q	P	17	XVII
r	K	1	II
s	M	11	V
t	I	20	IX

Parent Function Charades

New: 08/15/08

Objective:
Students will demonstrate an understanding of parent functions and transformations using function notation.

Connections to Previous Learning:
Students should be familiar with parent functions and transformations of functions using $f(x)$ notation: $f(x)+c$, $f(x+c)$, $cf(x)$, $-f(x)$, $f(-x)$.

Connections to AP*:
AP Calculus Topics: Analysis of Functions

Materials:
Student activity pages are formatted so they can be printed on business cards (2.5″ × 3″).
The "Parent Function" and "Transformation" pages are intended to be copied on the backs of the playing cards to differentiate the two types of cards.

Teacher Notes:
This activity can be done in teams or with the entire class. Decide which team (student) will go first. Set a maximum time limit for each turn.

To play the game, choose a person to:

- Draw one card each from the Parent Function and Transformation stacks.
- Act out the parent function.
- Act out the transformation(s). (When multiple transformations are required, player should act out changes separately.)
- Players turn ends when the team correctly identifies the transformed equation or the time limit expires.

PARENT | PARENT

FUNCTION | FUNCTION

PARENT | PARENT

FUNCTION | FUNCTION

PARENT | PARENT

FUNCTION | FUNCTION

PARENT | PARENT

FUNCTION | FUNCTION

PARENT | PARENT

FUNCTION | FUNCTION

$f(x)$ is the

constant

parent function

$f(x)$ is the

linear

parent function

$f(x)$ is the

absolute value

parent function

$f(x)$ is the

quadratic

parent function

$f(x)$ is the

square root

parent function

$f(x)$ is the

cubic

parent function

$f(x)$ is the

reciprocal

parent function

$f(x)$ is the

inverse square

parent function

$f(x)$ is the

exponential growth

parent function

$f(x)$ is the

exponential decay

parent function

$f(x)$ is the

logarithmic

parent function

$f(x)$ is the

natural logarithmic

parent function

$f(x)$ is the

sine

parent function

$f(x)$ is the

cosine

parent function

$f(x)$ is the

tangent

parent function

$f(x)$ is the

cotangent

parent function

$f(x)$ is the

secant

parent function

$f(x)$ is the

cosecant

parent function

$f(x)$ is the

arc cosine

parent function

$f(x)$ is the

arc sine

parent function

TRANSFORMATION TRANSFORMATION

TRANSFORMATION TRANSFORMATION

TRANSFORMATION TRANSFORMATION

TRANSFORMATION TRANSFORMATION

TRANSFORMATION TRANSFORMATION

$$-f(x) \qquad f(-x)$$

$$f(x)-4 \qquad f(x)+2$$

$$f(x-1) \qquad f(x+3)$$

$$-f(x+2) \qquad f(-x)-2$$

$$f(x+1)+2 \qquad f(x-3)+1$$

$$-\frac{1}{2}f(x)$$

$$2f(-x)$$

$$2f(x)+1$$

$$-\frac{1}{2}f(x)+2$$

$$f(x-2)$$

$$2f(x-1)$$

$$-f(x-3)$$

$$f(-x)+1$$

$$f(x-1)+3$$

$$f(x-2)-1$$

VERBAL DESCRIPTION VERBAL DESCRIPTION

VERBAL DESCRIPTION VERBAL DESCRIPTION

VERBAL DESCRIPTION VERBAL DESCRIPTION

VERBAL DESCRIPTION VERBAL DESCRIPTION

VERBAL DESCRIPTION VERBAL DESCRIPTION

Reflect across the x–axis

Reflect across the y–axis

Translate down 4

Translate up 2

Translate right 1

Translate left 3

Translate left 2 then
reflect across the x–axis

Reflect across the y–axis
then translate down 2

Translate left 1
then translate up 2

Translate right 3
then translate up 1

Half the y-values (vertical shrink) and reflect across x-axis

Reflect across the y-axis then double the y-values (vertical stretch)

Double the y-values (vertical stretch) then translate up 1

Half the y-values (vertical shrink); Reflect across the x-axis then translate up 2

Translate right 2

Translate right 1 then double y-values (vertical stretch)

Translate right 3 then reflect across the x-axis

Reflect across the y-axis then translate up 1

Translate right 1 then translate up 3

Translate right 2 then translate down 1

Parent Function Charades

Answers:

Answers will vary with cards drawn.

Activity 1: Paper Folding

Fold a piece of paper in half. Fold it in half again. Continue folding, filling in the table below.

1.

Number of folds	Process	Number of Layers of Paper
0		
1		
2		
3		
4		
5		
6		
n		

2. Write a function for the number of layers of paper you will have if you fold the paper n times.

3. Find a viewing window for the problem situation.

Sketch your graph: Note your window:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

4. Justify your window choices.

Use the home screen, graph, and table to find the following:

5. If you fold the paper 18 times, how many layers of paper will you have? Write the equation. Show how you got your solution.

6. A box of paper is 5 reams of paper deep. A ream of paper has 500 sheets of paper. About how many folds would you need to be at least as thick as a box of paper? Show how you found your solution.

Activity 2: Measure with Paper

A ream of paper measures approximately 2 inches thick.

1. If a ream is 500 sheets of paper, approximately how thick is a piece of paper?

2. Folding paper again, build the table and find a model.

Number of folds	Process	Thickness (inches)
0		
1		
2		
3		
4		
5		
6		
n		

3. Write a function for how thick the stack will be, in inches, if you fold the paper n times.

4. Find a viewing window for the problem situation.

Sketch your graph: Note your window:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

5. Justify your window choices.

Use the home screen, graph, and table to find the following:

6. If you fold the paper 15 times, how many inches of paper will you have? Compare this measurement to something in the room that has approximately the same measurement.

7. The Eiffel Tower is approximately 1050 feet tall. If you had a big enough piece of paper, how many folds would you need to match or exceed that height?

Activity 3: Regions

When you fold the piece of paper, you split the paper into regions, bounded by the fold lines. What fraction of the piece of paper is each region formed? Complete the table below.

1. Folding paper again, build the table and find a model.

Number of folds	Process	Fraction of the Piece of Paper
0		1
1		
2		
3		
4		
5		
6		
n		

2. Write a function for the fraction of a piece of paper for each region, if you fold the paper n times.

3. Find a viewing window for the problem situation.

Sketch your graph:

Note your window:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

4. Justify your window choices.

Use the home screen, graph, and table to find the following:

5. If you fold the paper 9 times, what fraction of the piece of paper is each region? Write your answer as a fraction. Give an example of a different situation where that fraction might appear.

6. Your school has a paper confetti machine that cuts 8.5” by 11” sheets of paper into about 400 pieces. What is the least number of times you need to fold the paper to get regions that are no larger than $\frac{1}{400}$ of the piece of paper?

Activity 4: How Big is a Region?

A piece of paper typing paper measures 8.5" by 11" inches.

1. What is the area in inches^2 of a piece of typing paper?

2. Folding paper again, build the table and find a model.

Number of folds	Process	Area of a Region
0		
1		
2		
3		
4		
5		
n		

3. Write a function for the area of a region, in inches^2, if you fold the paper *n* times.

4. Find a viewing window for the problem situation.

Sketch your graph:

Note your window:

*X*min:
*X*max:
*X*scl:
*Y*min:
*Y*max:
*Y*scl:

5. Justify your window choices.

Use the home screen, graph, and table to find the following:

6. If you fold the paper 10 times, what is the area of a region? Compare this measurement to something in real life that has approximately the same measurement.

7. Some plant cells have an area of approximately 1.55×10^{-5} in^2. How many folds do you need to have a region with at least that small of an area?

Activity 1: Exponential Growth

Suppose you have $1000 to invest. To simplify the comparisons, consider only interest compounded annually.

A. Your credit union's savings account is offering 4% interest.
B. The corner bank's savings account is offering 3% interest.
C. A Certificate of Deposit (CD) is offering 6.5% interest.

1. Write a function for each offer.
   A.
   B.
   C.

2. Compare the three offers in a table.

1 year			
2 years			
5 years			
10 years			
20 years			
30 years			

3. Compare the three offers graphically.

The M&M function
(Exponential Growth and Decay)

Members in your group (groups of two): ______________________________

Items Needed: one 1.69 ounce package of M&Ms per student (about 55 M&Ms)
one 16-ounce plastic Dixie Cup for each group

STEP 1:

Open your package of M&Ms. Place two of them in the cup at trial "zero." Roll them on the desk. For each *M* that is displayed, add one M&M. Then record the total number in the table for trial "one." Repeat until you reach 10 trials or 60 M&Ms, whichever comes first.

Trial #	M&Ms
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Do not eat the M&Ms .. we still need them!

STEP 2:

Now graph your results on the graph paper provided.

STEP 3:

Use your calculator to determine the exponential function which describes your data (see instructions below for using your calculator).

Write ***your*** exponential function here: ______________________________

What is the ***theoretical*** exponential function for this experiment? ______________________________

STEP 4:

Graph the points in your table on the calculator. Then graph your exponential function on the calculator. It should pass through most of the points that you plotted. Now graph the theoretical exponential function on your calculator.

See instructions below for graphing.

STEP 5:

Start with the number of M&Ms in your Dixie cup that you finished with in STEP 1. Put this number in trial "zero." Now roll out the M&Ms on your desk. Any that show an ***M*** get removed (and eaten ☺). Count the remaining M&Ms and write this number in trial "one." Repeat for 10 trials or until all the M&Ms are gone.

Trial #	M&Ms
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

STEP 6:

Now graph your results on the graph paper provided.

STEP 7:

Use your calculator to determine the exponential function which describes your data (see instructions below for using your calculator).

Write ***your*** exponential function here: ______________________________

What is the ***theoretical*** exponential function for this experiment? ______________________________

STEP 8:

Graph the points in your table on the calculator. Then graph your exponential function on the calculator. It should pass through most of the points that you plotted. Now graph the theoretical exponential function on your calculator.
See instructions below for graphing.

Click here for graph paper

Click here for instructions for the TI-83 Plus Calculator

Click here for instructions for the TI-85 Calculator

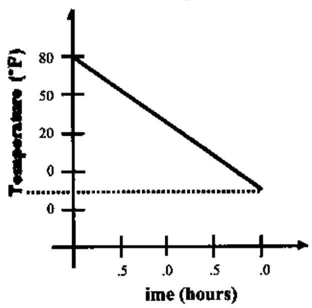
Click here for instructions for the TI-86 Calculator

Activity: Cooling Down

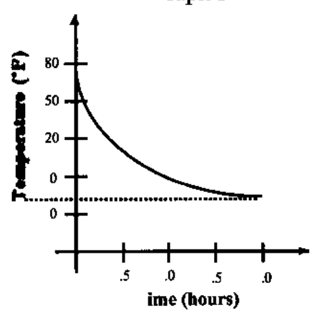
Consider a cup of hot water at approximately 180° F that is placed in a room where the temperature is 74° F. Think about the cooling effect.

1. How will the water cool over time?

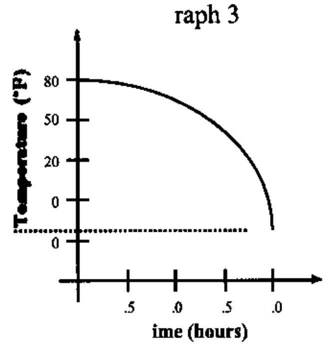
raph 1



raph 2



raph 3



2. Describe a situation implied by each graph.

Use a thermometer:

We will simulate the water cooling down by recording the temperature data of a thermometer cooling down.

Group Roles:
Temperature reader – handles the thermometer and calls out the temperature.
Timer – keeps track of the elapsed time.
Recorder – records the data.

3. Use the thermometer to determine the room temperature. ________
4. Pour hot, not boiling water ($\approx 180°$ F) into a cup. Place the thermometer in the cup for about a minute to heat up.

5. Predict the number of seconds that you think it will take your thermometer to "cool down" to room temperature. ________
6. Remove the thermometer from the cup of hot water. (Make sure to move the thermometer away from the steam of the cup.) Begin collecting data immediately.
   Although the independent variable is time, you may find it easier to call out temperature decreases of 2° and record the elapsed time. The data recorder is responsible for recording all data. Be ready for quick readings initially.

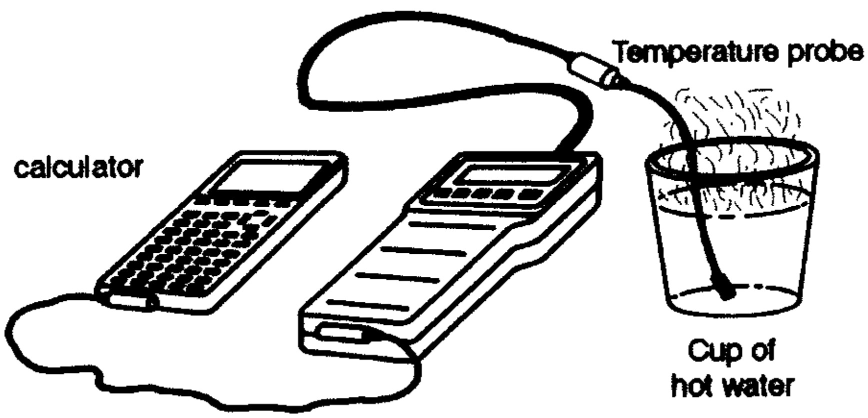
Elapsed Time (seconds)	Temperature (°F or °C)

7. What patterns do you see in the data?

Use a graphing calculator and an electronic data collection device with a temperature probe.

We will simulate the water cooling down by recording the temperature data of a temperature probe cooling down.

Group Roles:
Temperature reader - handles the temperature probe.
Calculator – handles the graphing calculator.
Recorder – handles the electronic data collection device.



3. Use the temperature probe to determine the room temperature. _______

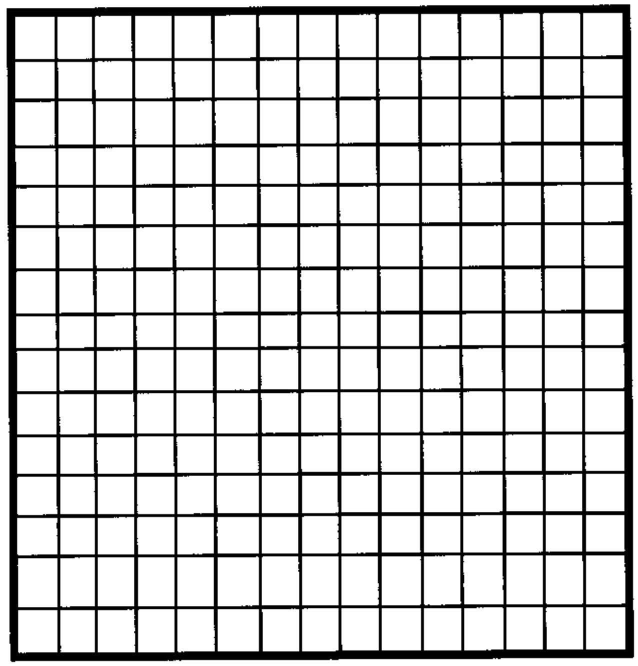
4. Pour hot, not boiling water (≈ 180° F) into a cup. Place the temperature probe in the cup for about a minute to heat up.

5. Predict and record the number of seconds that you think it will take your thermometer to "cool down" to room temperature.

6. Remove the thermometer from the cup of hot water. (Make sure to move the thermometer away from the steam of the cup.) Begin collecting data immediately.

7. What patterns do you see in the data?

8. Sketch a scatter plot of the data collected on the grid below.



9. What family of functions has the characteristics seen in the graph of the data? What is different about the graph of the data and the general function in this family?

10. Adjust a list of data collected to create a scatter plot with a horizontal asymptote at $y = 0$. Use a graphing calculator to find an exponential regression equation of the form $y = ab^x$ that best fits the data.

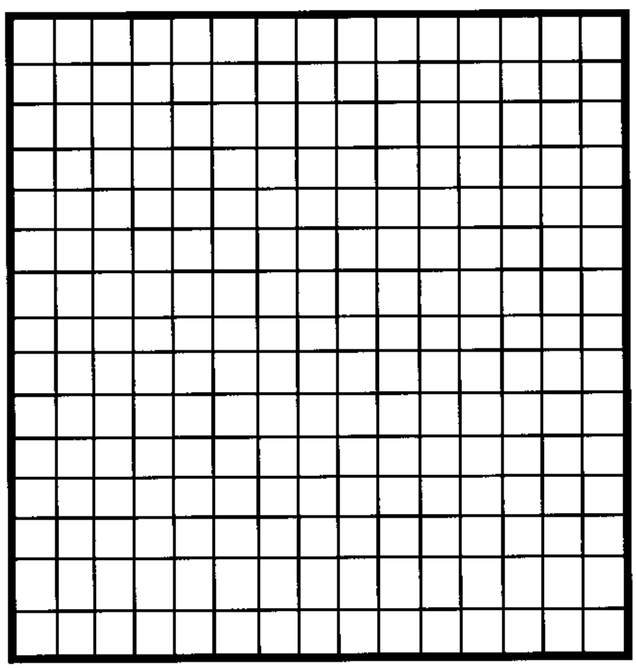
11. Does this regression equation model the actual data? If not, how can you adjust it so that it does?

Reflect and Apply

1. Convert the °C to °F by creating a new list L_4 that is equal to $1.8 * L_2 + 32$.
   Record findings.

2. How are the two graphs related to each other. Why?

3. Create a scatter plot of L_1, L_4.
   Select a window that will accommodate both the original scatter plot and the new scatter plot. Sketch the plots below.



4. Compare and contrast the two graphs. Do the two graphs run parallel? If so, why? If not, why not?

Day 5: Penny Problem

(Taken from Jeanne Campion's binder of lessons)

Objectives:

To abstract the idea of a line
To introduce other methods of solving simultaneous equation besides graphing
To develop problem solving skills

Materials:

A scale able to read (up to the nearest tenth of a gram) up to 400 grams at least 1,000 pennies
A plastic cup to hold pennies on the scale
Numbered plastic ziplock bags 1-8
16 index cards, 2 per ziplock bag

Activity:

Place the 1,000 pennies in a large pile on the front desk along with the scale, the plastic cup and ziplock bags. Divide the class into 8 groups. From each group have one member come up and take a handful of pennies. Weigh and record the amount 'W'. Place the weighed pennies in the corresponding ziplock bag and give the group member the instructions to separate the pennies into two groups: 1981 and before and label as 'B', 1983 and after label as 'A'. They are to record the number of each kind of penny on one of the index cards provided and keep it from your view. On the other index card, they are to write the total number of pennies in the bag 'T'. Place this second index card and all pennies in the plastic bag and return to front desk. As the bags are returned record the number of pennies and calculate the predicted number of pennies for 1982 and before for each group. When all the pennies have been returned and the calculations finished, announce that through the miracle of algebra you know how many of each type they have sorted with a margin of error plus or minus two pennies. Explain to the class why you are correct.

Pennies for 1981 and before weigh 3.1 grams
Pennies for 1983 and after weigh 2.5 grams
(Note: avoid 1982 pennies because the mint produced some of each)

	Number	Weight	Total Weight
1981 and before	B	3.1	3.1B
1983 and after	A	2.5	2.5A
Total	T		W

Which gives us the equations:

$$3.1B + 2.5A = W$$
$$B + A = T$$

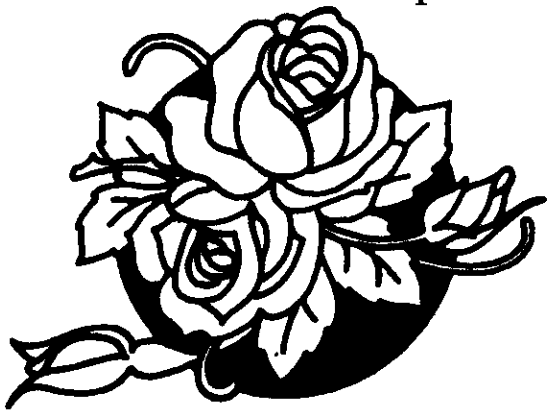
The problem becomes just finding the intersection of two lines

Solving by substitution

Assessment: Finish any work on Penny Problem

Activity 1: Valentine's Day Idea

The school's drill team has contacted several flower distributors and has narrowed the choice to two companies.



Option 1: **Roses-R-Red** has offered to sell its roses for a fixed down payment of \$20 and an additional charge of 75 cents per stem.

Option 2: **The Flower Power** has offered to sell its roses for a fixed down payment of \$60 and an additional charge of 50 cents per stem.

Which is the more economical offer?

Activity 2: Using Tables to Find the More Economical Offer

From the description of the two offers, complete the chart to find an algebraic rule that will determine the cost of *n* roses.

Number of Roses	Process Column (Roses-R-Red)	Cost at Roses-R-Red	Process Column (Flower Power)	Cost at Flower Power
10				
20				
30				
60				
90				
120				
150				
180				
210				
240				
1000				
n				

1. Write a sentence and a function rule for the cost of roses from Roses-are-Red.

2. Write a sentence and a function rule for the cost of roses from Flower Power.

3. What patterns do you observe from the table of values?

4. What happens to the cost of the roses as the number of roses purchased increases? What would a graph of this relationship look like?

5. How many roses can you buy from Roses-are-Red for $65.00?

6. How many roses can you buy from Flower Power for $65.00?

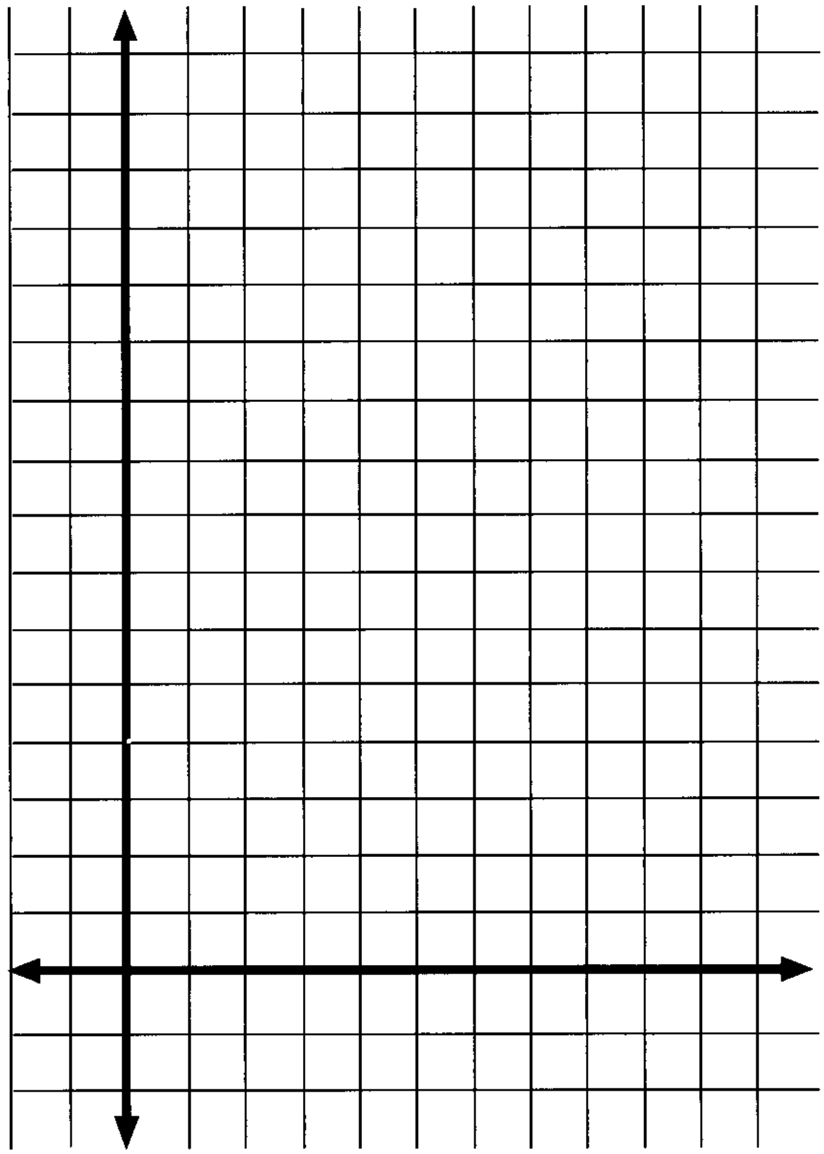
7. Which company offers the better deal?

8. Is there a point where the two flower dealers charge the same total amount? If so, what is the charge? If not, why do the costs never equal?

9. Write an equation that represents the point where the two flower shops charge the same amount.

Activity 3: Using Graphs to Find the Better Offer

1. Find an appropriate viewing window for the graphs of both functions. Sketch both functions here and label.



2. Justify your viewing window choice:
   *x*min:
   *x*max:
   *y*min:
   *y*max:

3. What effect does the 75 cents per stem cost have on the graph of the Roses-R-Red function? What effect does the $20 have on the graph?

4. What effect does the 50 cents per stem cost have on the graph of the Roses-R-Red function? What effect does the $60 have on the graph?

5. What are the coordinates of the point of intersection of the two functions? What is the significance of this point?

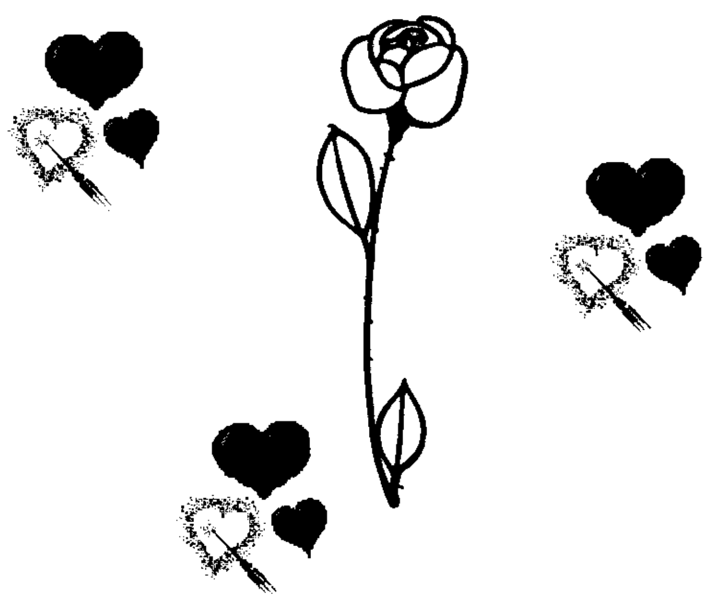
6. Which flower dealer offers the better deal? Justify your answer.

Activity 4: New Rose Offers

To entice these potential new customers, Roses-R-Red decides to eliminate its fixed charge of $20. According to its new offer, the drill team pays only for the roses they buy. When the Flower Power learns about the new offer by its competitor, it immediately enters the price war by reducing its fixed charge also by $20.

Which new deal is the better offer?

How does the new offer compare to the original offer?



Activity 5: Using Tables for New Rose Offers

From the description of each of the two new deals, complete the chart and write new algebraic rules that will determine the cost of n roses.

Number of Roses	Process Column (Roses-R-Red)	Cost at Roses-R-Red	Process Column (Flower Power)	Cost at Flower Power
10				
20				
30				
60				
90				
120				
150				
180				
210				
240				
270				
300				
n				

1. What patterns do you observe in the new table of values?
2. Compare the costs on this chart to the costs on the first chart. What changes do you observe? Predict what the graphs will look like.
3. Which company offers the better deal?
4. Is there a point where the two flower dealers charge the same amount? If so, what is the charge?
5. Write an equation that represents the point where the two flower shops charge the same amount.

Activity 6: Using Graphs for New Rose Offers

1. On your graphing calculator graph only the original Roses-R-Red function, $y = 20 + 0.75x$. Now, predict what you think the graph of the new offer, $y = 0.75x$, will look like.

2. Graph both the original offer, $y = 20 + 0.75x$, and the new offer, $y = 0.75x$, together.

3. What effect does subtracting \$20 from the old rule have on the new graph of the Roses-R-Red function?

4. Turn off the above two graphs. Graph only the original Flower Power function, $y = 60 + 0.5x$. Now, predict what you think the graph of the new offer will look like, $y = 40 + 0.5x$.

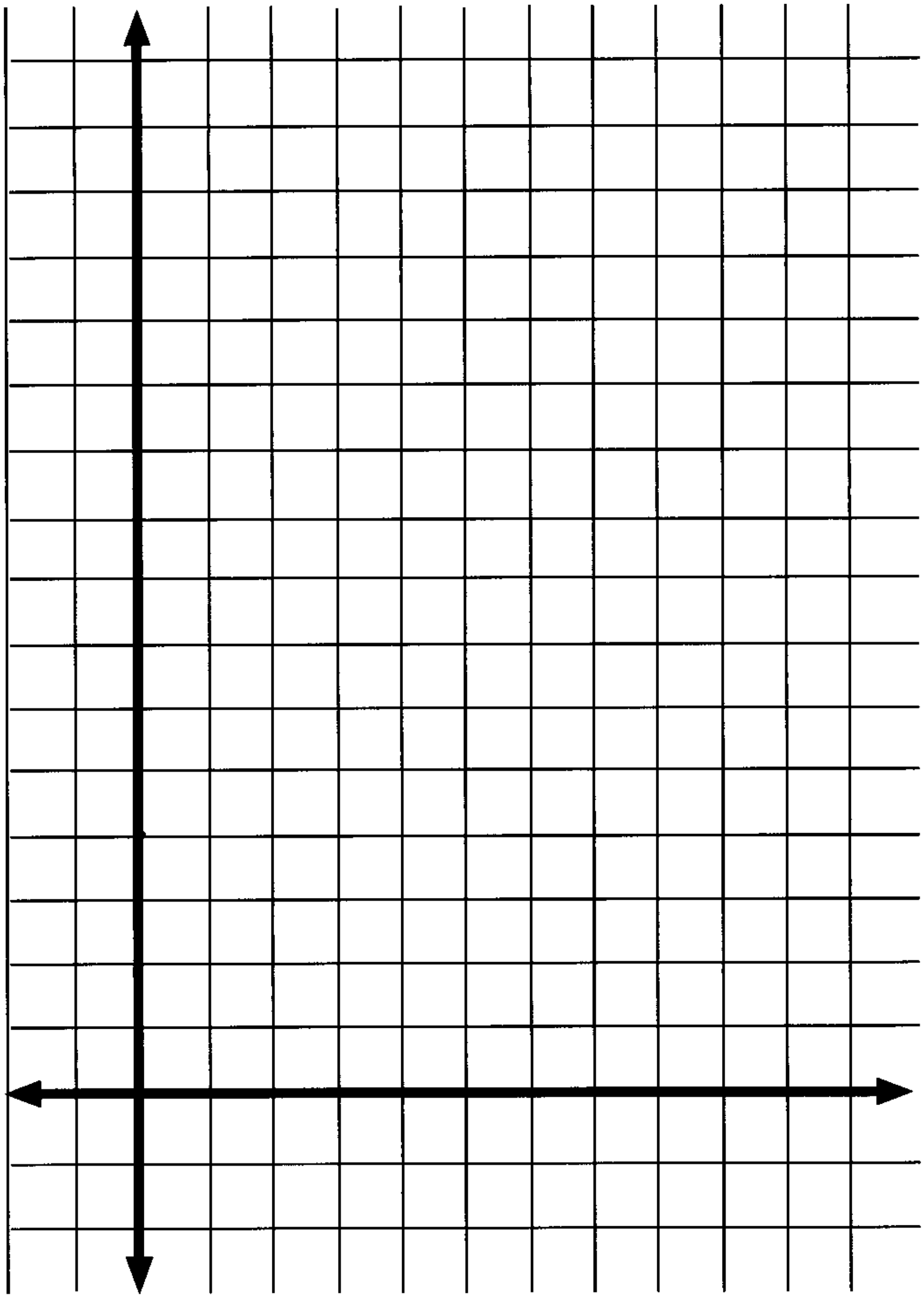
5. Graph both the original offer, $y = 60 + 0.5x$, and the new offer, $y = 40 + 0.5x$, together.

6. What effect does subtracting \$20 from the old rule have on the new graph of the Flower Power function?

7. Graph all four functions at the same time. What are the coordinates of the point where the two new functions intersect? What is the significance of this point?

8. Which flower dealer now offers the better deal?

9. Sketch the graphs of all four functions and label the relevant points of intersection.



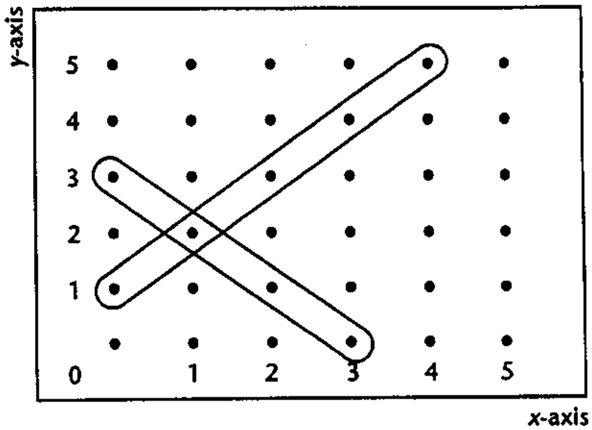
NAME ______________________ CLASS ____________ DATE ____________

Lesson Activity

9.1 Geoboard Systems

You can use a geoboard as a coordinate plane. Use the bottom of the geoboard as the x-axis and the left-hand side of the geoboard as the y-axis. You can then use rubber bands to show graphs of linear equations. Part of a geoboard on which the equations $y = x + 1$ and $y = -x + 3$ have been graphed is shown to the right. As you can see, the "coordinates" of the peg that both rubber bands contain is (1, 2). So the solution to the system of equations

$\begin{cases} y = x + 1 \\ y = -x + 3 \end{cases}$ is (1, 2).



Use a geoboard to solve each system of equations. Check your solution.

1. $\begin{cases} y = x + 1 \\ y = -2x + 4 \end{cases}$ ____________

2. $\begin{cases} y = x + 2 \\ y = -2x + 5 \end{cases}$ ____________

3. $\begin{cases} y = 2x \\ y = -x + 3 \end{cases}$ ____________

4. $\begin{cases} y = 2x + 1 \\ y = -x + 4 \end{cases}$ ____________

5. $\begin{cases} y = 3x \\ y = 3 \end{cases}$ ____________

6. $\begin{cases} y = -\frac{1}{3}x + 4 \\ y = x \end{cases}$ ____________

7. $\begin{cases} y = \frac{1}{5}x \\ y = -\frac{4}{5}x + 5 \end{cases}$ ____________

8. $\begin{cases} y = 0.5x + 1 \\ y = -x + 4 \end{cases}$ ____________

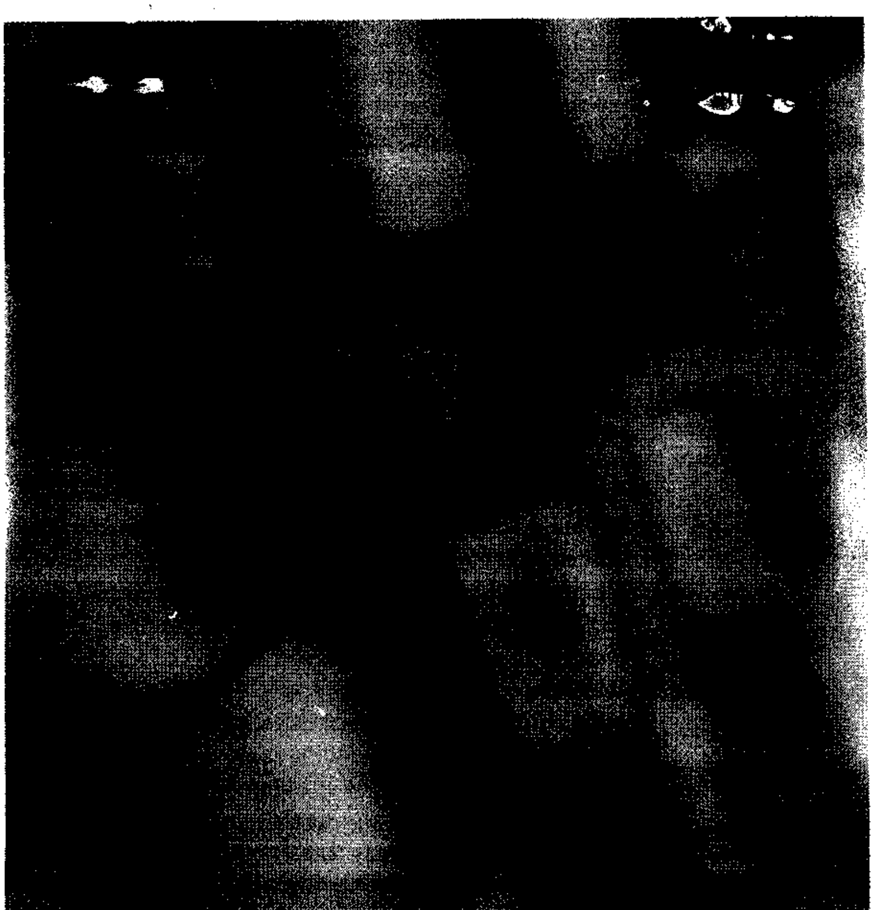
9. Write a system of your own that has a solution of (3, 3) and that can be solved by using a geoboard. ____________

10. Write a system of your own that has a solution of (0, 4) and that can be solved by using a geoboard. ____________

11. Write a system of your own that has a solution of (2, 5) and that can be solved by using a geoboard. ____________

Solving Simultaneous Equations: Getting More from Geometry

Gerald Gannon and Harris S. Shultz



Consider the student when he/she first encounters the problem of solving a system of two equations in two unknowns, such as:

$$\begin{aligned} 3x + 2y &= 18 \\ 5x - y &= 17 \end{aligned} \qquad (1)$$

The student knows that each of these equations represents a straight line, and a graph of this situation (**fig. 1**) is then used to illustrate the fact that these two lines should intersect exactly once. Although the geometry motivates the existence of a solution, the student quickly reverts to an "algebra" mode of thinking. In this case, the student would most likely solve the second equation for y and then substitute that expression for y in the first equation, obtaining

$$3x + 2(5x - 17) = 18.$$

The student would now solve this equation, obtaining $x = 4$. Substituting this value into the equation $y = 5x - 17$, y is found to be 3, thus giving the final answer of (4, 3).

If none of the variables started with a coefficient of 1, the student would probably then be prompted to solve the system using the process of "elimina-

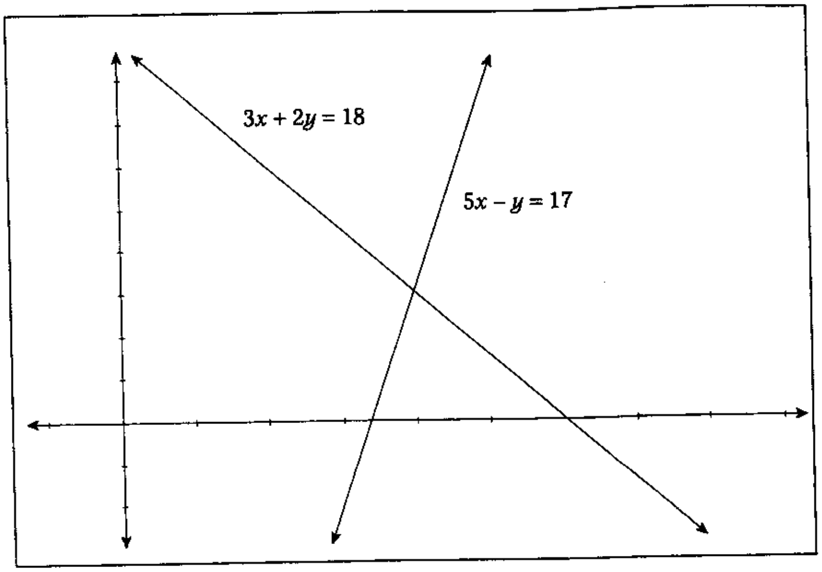


Fig. 1 Graph of the system (1)

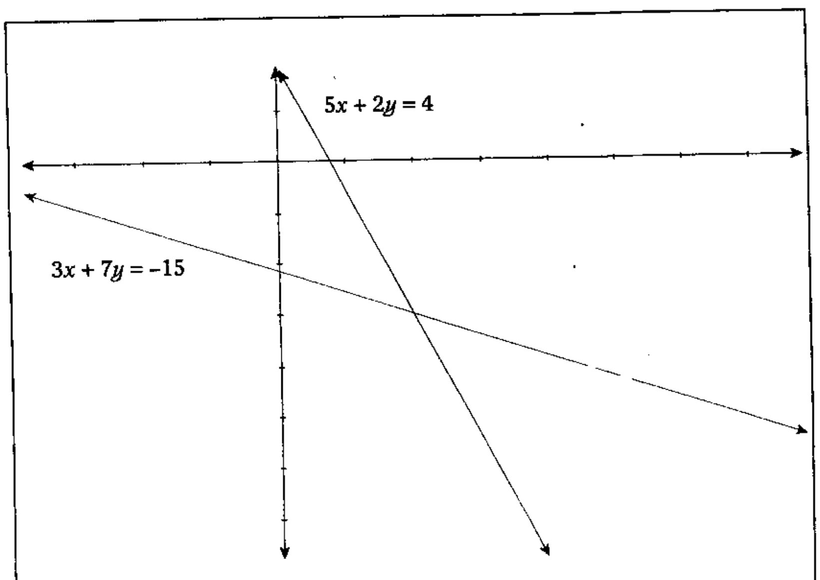


Fig. 2 Graph of the system (2)

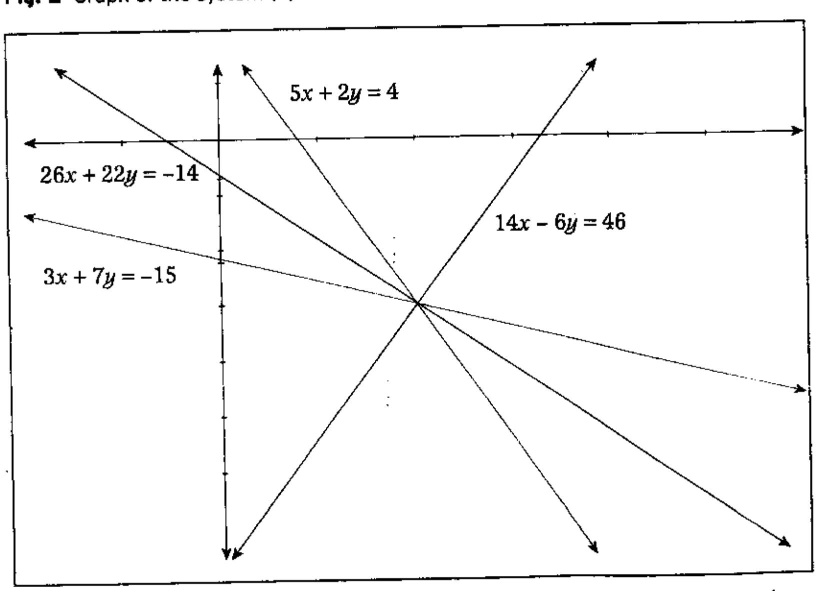


Fig. 3 Graph of system (2) and two linear combinations of the equations of the system

tion." For example, consider

$$(2)\qquad \begin{aligned} 5x + 2y &= 4 \\ 3x + 7y &= -15. \end{aligned}$$

Here the student would most likely multiply the first equation by 3 and the second equation by –5 to obtain the system

$$\begin{aligned} 15x + 6y &= 12 \\ -15x - 35y &= 75 \end{aligned}$$

The student may well have just been told that multiplying by these constants will not affect the solution. Now that the coefficients of x are opposites, the left and right sides are added, yielding $-29y = 87$, or $y = -3$. This value of y would then be substituted into either of the two original equations to find that $x = 2$. Hence, the solution: $(2, -3)$.

Although this approach does indeed allow students to solve a system of equations, the authors believe it robs them of seeing some of the beauty of mathematics by denying them the experience of understanding the geometry of what they are doing. Of course, geometry is mentioned briefly when discussing what it means for the system to be dependent or inconsistent. Let's go back to (2). The graph of this system is shown in **figure 2**. This time, let's multiply each equation by some "inconvenient" numbers so that neither x nor y ends up with the same or opposite coefficients. For example, multiply the first equation by 4 and the second equation by –2, obtaining the system

$$\begin{aligned} 20x + 8y &= 16 \\ -6x - 14y &= 30. \end{aligned}$$

Certainly, if the students have never done so, they should graph this system to verify for themselves that nothing about the graph has changed by multiplying through an equation by a constant. Next, have students add these two equations, obtaining $14x - 6y = 46$. While they are at it, have them subtract the second equation from the first, obtaining $26x + 22y = -14$. Next, have them graph these two new equations on the same grid they had graphed (2) (see **fig. 3**). They should notice that although they get two different lines, both of the new lines pass through the point of intersection of the original system (**fig. 2**).

By doing several such activities, students will come to see that linear combinations of the original equations will produce an equation that passes through the point that represents the solution to the original system. If any line so obtained still goes through the point whose coordinates one is interested in finding, are there any lines that make more

294

sense to use than others? With such a prompt, students should soon come to the conclusion that horizontal and vertical lines through the intersection point would make finding the coordinates of that point quite easy. But what do the equations of horizontal and vertical lines look like? They contain only one variable, either x or y. Hence, if one could add the two equations in such a way as to produce an equation with only one variable in it, one would have found either the horizontal or vertical line going through the point being sought. When we manipulate to discover for (2) that $x = 2$ and $y = -3$, the geometric reality is that the student has found the equations of the two easiest lines to work with, for it is a simple task to determine where these two lines intersect, namely, at the point $(2, -3)$.

We conclude by following through with the complete solution of (2), stopping to observe the associated graph at each step. Recall that the graph of (2) is shown in **figure 2**. In order to eliminate y, we multiply the first equation by 7 and the second equation by -2:

$$(3) \qquad \begin{aligned} 35x + 14y &= 28 \\ -6x - 14y &= 30 \end{aligned}$$

Now, replace the second equation in (3) with the sum of the two equations:

$$(4) \qquad \begin{aligned} 35x + 14y &= 28 \\ 29x &= 58 \end{aligned}$$

Dividing the first equation by 7 and the second equation by 29 simplifies (4) to

$$(5) \qquad \begin{aligned} 5x + 2y &= 4 \\ x &= 2. \end{aligned}$$

In **figure 4** we see the graph of (5) and we observe that one of the lines is vertical. In order to eliminate x we multiply the second equation by -5:

$$(6) \qquad \begin{aligned} 5x + 2y &= 4 \\ -5x &= -10 \end{aligned}$$

Now, replace the first equation in (6) with the sum of the two equations:

$$(7) \qquad \begin{aligned} 2y &= -6 \\ -5x &= -10 \end{aligned}$$

Equivalently, we can write

$$(8) \qquad \begin{aligned} y &= -3 \\ x &= 2. \end{aligned}$$

The graph of (8), clearly showing the solution, is given in **figure 5**.

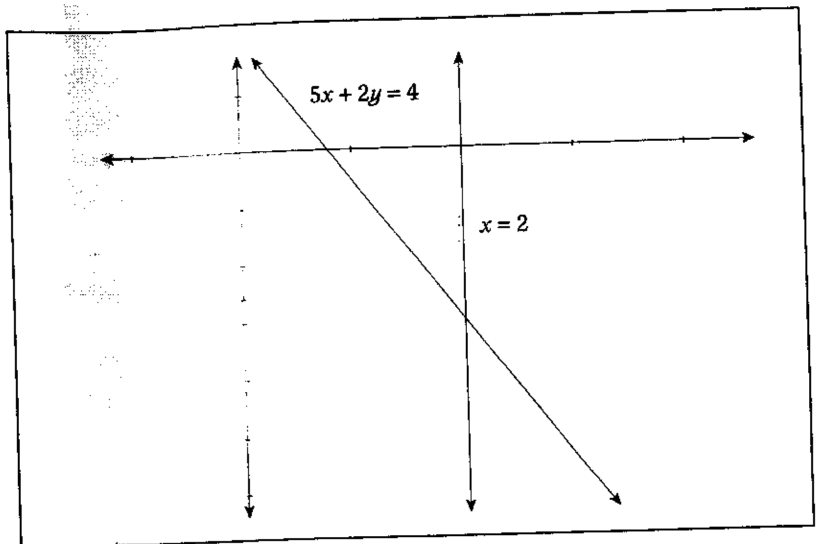


Fig. 4 The vertical line gives the x-coordinate of the solution to the system.

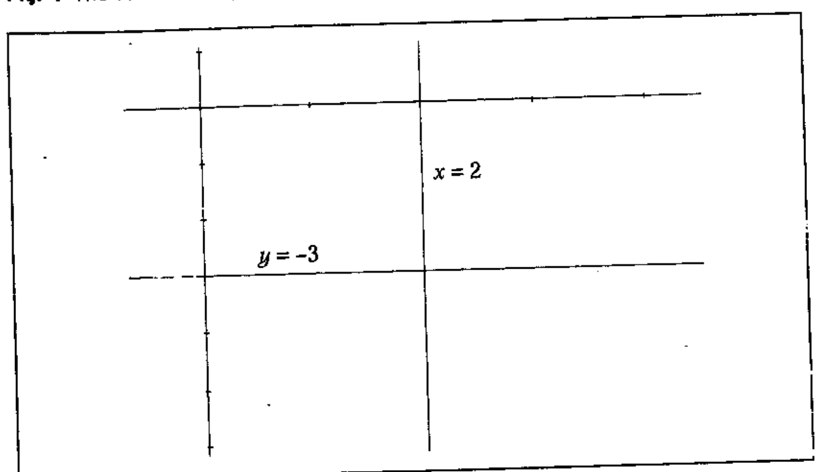


Fig. 5 Vertical and horizontal lines give the complete solution to the system.

In summary, having students graph some linear combinations of the original two lines might deepen their understanding of what is happening geometrically as they manipulate the algebra. Since we stress the connections between the equations and their graphs, why not use this connection to further deepen their understanding of the interactions between algebra and geometry? ∞

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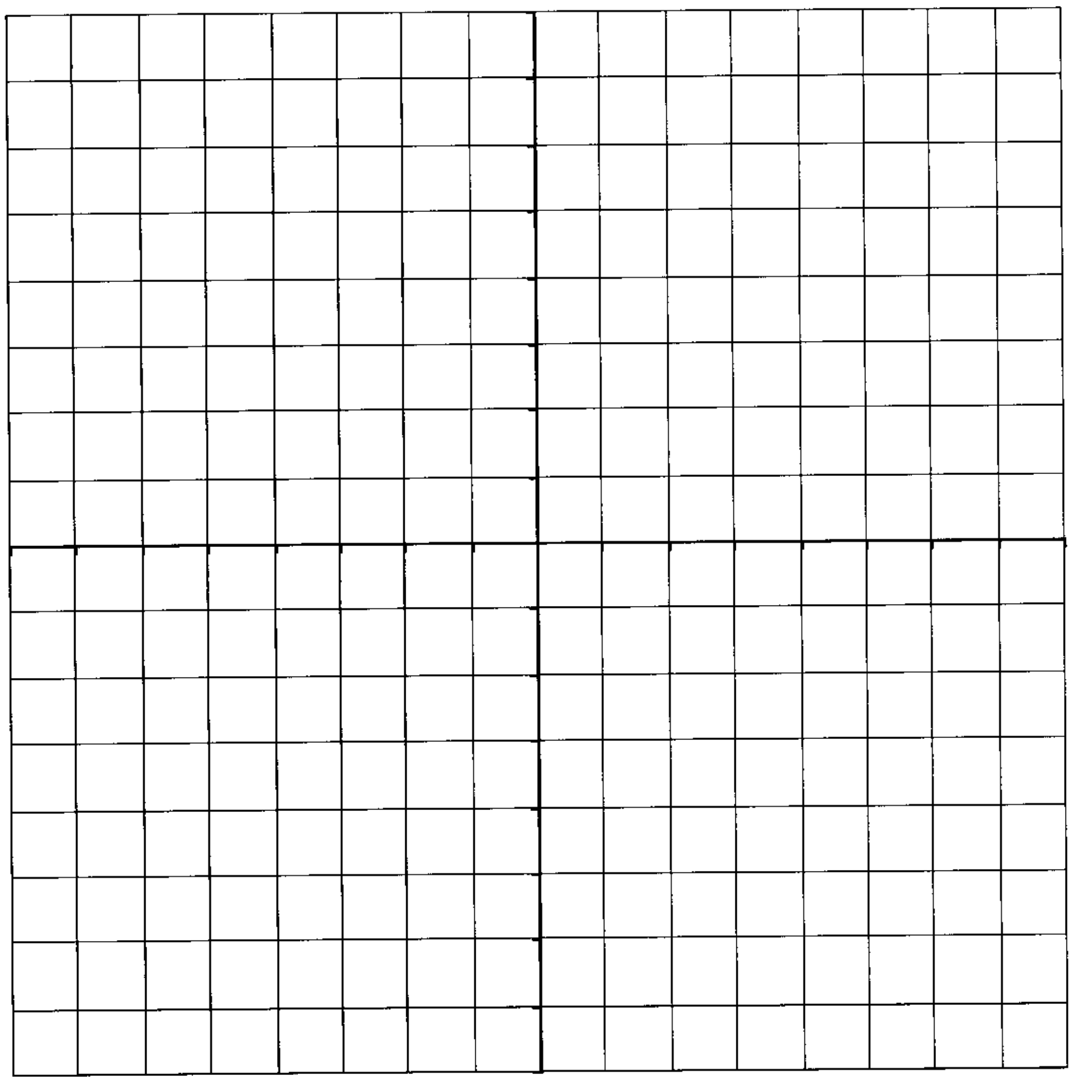
Activity 3: Inverses

1. Graph $y = x^2$ and $y = x$ on the grid. Place the MIRA along the line $y = x$ and sketch the reflection of $y = x^2$.



2. Make a table that shows values for the original function and corresponding values for the reflected function. Describe patterns.

3. Is the reflection a function? Why or why not?

4. What could you do to make the inverse a function?

Investigating the Inverses of Functions Using Patty Paper

Materials: 5 sheets of Patty Paper per person, one ruler per person, colored pencils

Investigation I:

1. Fold a sheet of Patty Paper twice to form the axes of a Cartesian plane. Label the x-axis and the y-axis.

2. Carefully fold your sheet of Patty Paper to form the line $y = x$. Label this line.

3. What are the characteristics of all the points that lie on the line $y = x$?

4. Sketch the function $y = 2x + 1$ on your Cartesian plane.

5. Fold the sheet of Patty Paper along the line $y = x$ to find the reflection of $y = 2x + 1$ across the line $y = x$. Sketch this image a different color from your sketch of $y = 2x + 1$.

6. Unfold the sheet of Patty Paper. Describe the relationship between the original graph and its reflection across the line $y = x$.

7. Make two tables of x- and y-values, one for $y = 2x + 1$ and one for its reflection across the line $y = x$. Describe the relationship between the two tables.

8. What are the x- and y-intercepts of $y = 2x + 1$ and its reflection across the line $y = x$? Label them on your Cartesian plane. Describe the relationships that you see.

9. Find the equation of this image of $y = 2x + 1$?

10. Is this image of $y = 2x + 1$ a function? Explain.

Investigation II:

1. Fold a sheet of Patty Paper twice to form the axes of a Cartesian plane. Label the x-axis and the y-axis.

2. Carefully fold your sheet of Patty Paper to form the line $y = x$. Label this line.

3. Sketch the function $y = x^2$ on your Cartesian plane.

4. Fold the sheet of Patty Paper along the line $y = x$ to find the reflection of $y = x^2$ across the line $y = x$. Sketch this image a different color from your sketch of $y = x^2$.

5. Unfold the sheet of Patty Paper. Describe the relationship between the original graph and its reflection across the line $y = x$.

6. Make two tables of x- and y-values, one for $y = x^2$ and one for its reflection across the line $y = x$. Describe the relationship between the two tables.

7. What are the x- and y-intercepts of $y = x^2$ and its reflection across the line $y = x$? Label them on your Cartesian plane. Describe the relationships that you see.

8. Find the equation of this image of $y = x^2$?

9. Is this image of $y = x^2$ a function? Explain.

Investigation III:

1. Fold a sheet of Patty Paper twice to form the axes of a Cartesian plane. Label the x-axis and the y-axis.

2. Carefully fold your sheet of Patty Paper to form the line $y = x$. Label this line.

3. Sketch the function $y = x^3$ on your Cartesian plane.

4. Fold the sheet of Patty Paper along the line $y = x$ to find the reflection of $y = x^3$ across the line $y = x$. Sketch this image in a different color from your sketch of $y = x^3$.

5. Unfold the sheet of Patty Paper. Describe the relationship between the original graph and its reflection across the line $y = x$.

6. Make two tables of x- and y-values, one for $y = x^3$ and one for its reflection across the line $y = x$. Describe the relationship between the two tables.

7. What are the x- and y-intercepts of $y = x^3$ and its reflection across the line $y = x$? Label them on your Cartesian plane. Describe the relationships that you see.

8. Find the equation of this image of $y = x^3$?

9. Is this image of $y = x^3$ a function? Explain.

Summary:

1. In the three investigations, which of the functions had images that were functions?

2. What would you have to do to a function whose image was not a function to make its image a function?

3. Describe the images of points on the original function that intersected the line $y = x$?

4. Repeat Investigation I for $y = 4$? Explain what you observe.

5. Explain the relationship between a function and its inverse. Your explanation should include descriptions of the relationships in tables, graphs, and equations.

6. Is the inverse of a function always a function? Explain.

7. Is the inverse of a function always a relation? Explain.

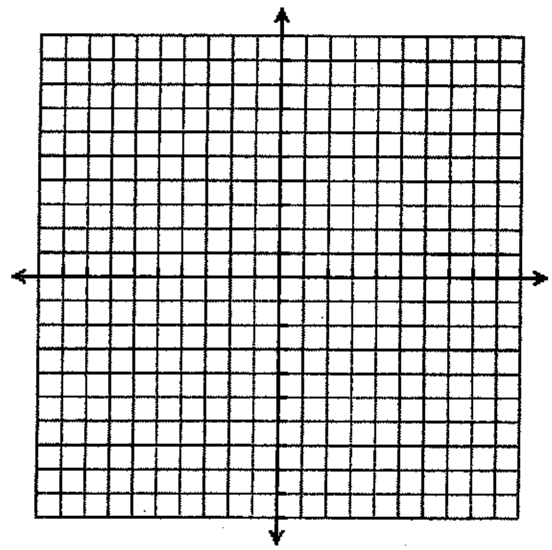
8. Name two functions that are their own inverses.

Name:_______________________________Date:_____________Period:____

Mirror, Mirror!

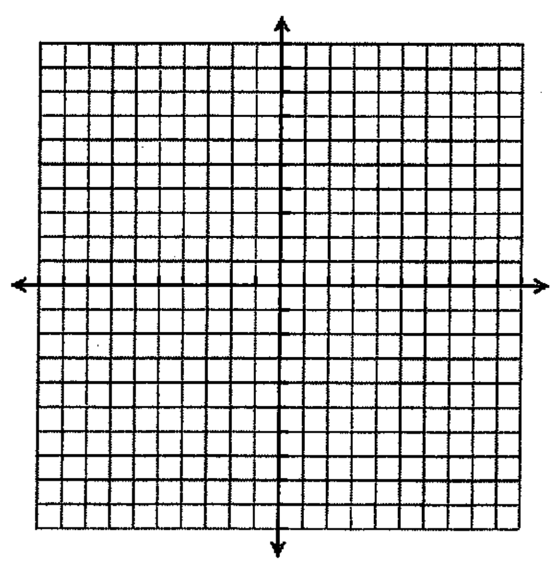
1. Using a standard window on your calculator, graph the equation $y = x^2$. Use the table function to fill in the chart below. Square the window ($\boxed{\text{ZOOM}}\boxed{5}$). Plot the points to draw the graph on the grid.

x	$y = x^2$
0	
1	
2	
3	
4	



2. Now graph the equation $y = \sqrt{x}$. Use the table function to fill in the chart below. Plot the points to draw the graph.

x	$y = \sqrt{x}$
	0
	1
	2
	3
	4



3. Examine the table in #1 and #2. What do you notice about the numbers in the table?

4. Graph $y = x$ and the two equations from above. You should see all three graphs on the window at the same time. ($y = x^2$, $y = \sqrt{x}$ and $y = x$). Your graphs ~~should look like the~~ figure at the bottom of Page 3.

5. Use a sheet of patty paper and neatly trace the x- and y-axis and the three graphs at the bottom of Page 3. Fold your patty paper along the y = x line. What do you notice about the two graphs? (Staple the patty paper below.)

6. Now graph $y = x^2 + 2$. Make a table like the one in # 1.

x	y
0	
1	
2	
3	
4	

7. Take the numbers from your table and reverse the numbers to make a new table.

x	y

8. Use your calculator with the numbers in the new table to make a scatter plot of the data. What do you think will be the equation of the curve that will fit this new data?

9. Use guess and check to find an equation that will fit the curve. Write your equation below.

10. Put this equation in the calculator and graph the three equations **in a square window** ($\boxed{\text{ZOOM}}\boxed{5}$):

Y1: $y = x^2 + 2$

Y2: your equation

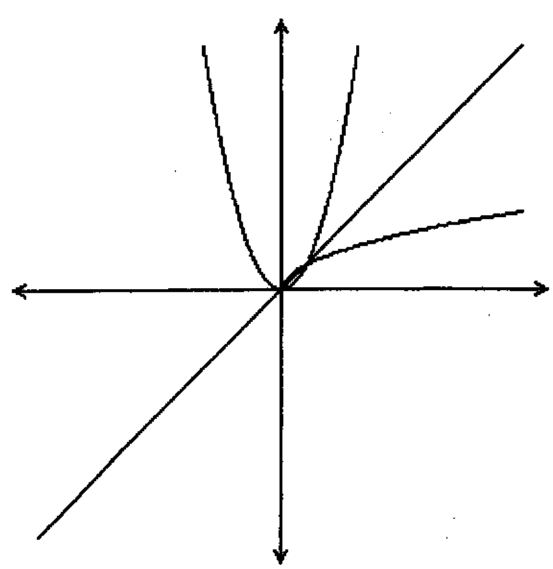
Y3: $y = x$

Place a sheet of patty paper on the calculator screen and trace your graphs. Fold along the y = x line. Describe the relationship between the first two graphs.

11. Does the second equation reflect the entire graph of the first equation?_____ What other equation do you think we could use to get the rest of the graph to reflect?

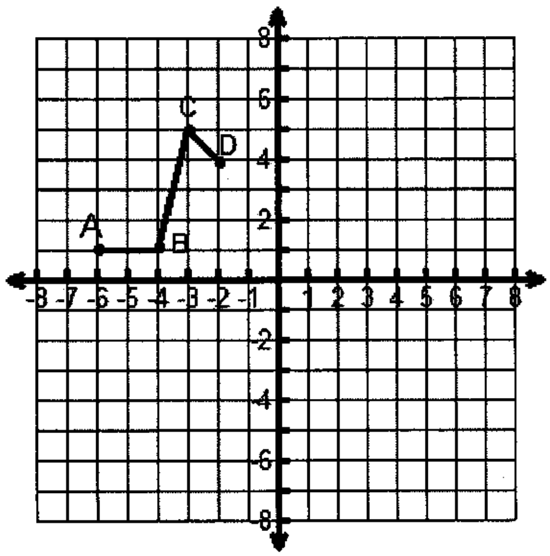
12. Now make up a quadratic equation of your own and write equations that will reflect the graph over the y = x line. Print your graphs out and attach them to your work. Show your equation and its inverse equation below.

Use the figure to the right to complete #5



Exploration of the Inverse of a Function

Let's call the graph below f(x).



1. Make a table listing the coordinates of the points on the given graph.

2. Trace f(x) on patty paper.
3. Sketch the reflection of f(x) over y = x (the identity function).
4. Label the new graph "inverse of f(x)".
5. Make another table listing the reflected points.

6. In your own words, describe how the points changed as they were reflected over the identity function.

7. What is the domain and range for f(x)? What is the domain and range for the inverse of f(x)?

8. In your own words describe how the domain and range for the two relations compare.

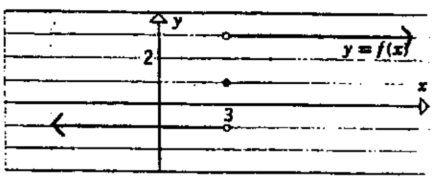
9. Is the reflected graph a function? If yes, we call f(x) a one-to-one function and the inverse $f^{-1}(x)$.

1.2 Practice

Name:

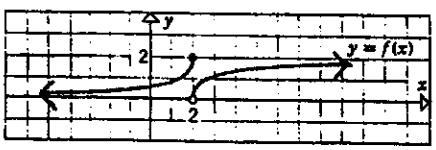
1. For the function f graphed to the right, find

(a) $\lim\limits_{x\to 3^-} f(x)$ (b) $\lim\limits_{x\to 3^+} f($
(c) $\lim\limits_{x\to 3} f(x)$ (d) $f(3)$
(e) $\lim\limits_{x\to -\infty} f(x) =$ (f) $\lim\limits_{x\to +\infty} f(x)$.



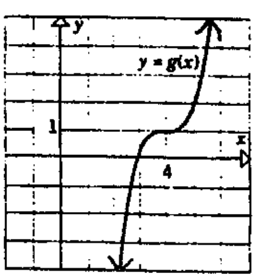
2. For the function f graphed below, find

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to 2^+} f(x)$
(c) $\lim\limits_{x\to 2} f(x)$ (d) $f(2)$
(e) $\lim\limits_{x\to -\infty} f(x)$ (f) $\lim\limits_{x\to +\infty} f(x)$.



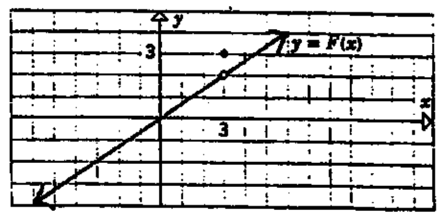
3. For the function g graphed below, find

(a) $\lim\limits_{x\to 4^-} g(x)$ (b) $\lim\limits_{x\to 4^+} g(x)$
(c) $\lim\limits_{x\to 4} g(x)$ (d) $g(4)$
(e) $\lim\limits_{x\to -\infty} g(x)$ (f) $\lim\limits_{x\to +\infty} g(x)$.



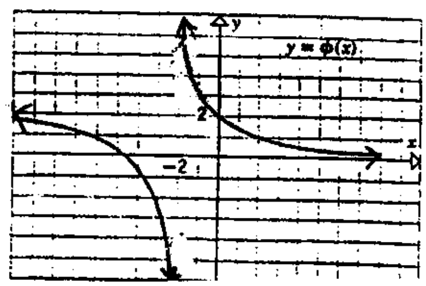
4. For the function F graphed below, find

(a) $\lim\limits_{x\to 3^-} F(x)$ (b) $\lim\limits_{x\to 3^+} F(x)$
(c) $\lim\limits_{x\to 3} F(x)$ (d) $F(3)$
(e) $\lim\limits_{x\to -\infty} F(x)$ (f) $\lim\limits_{x\to +\infty} F(x)$.



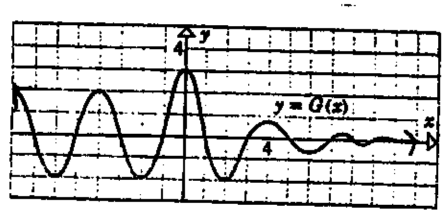
5. For the function ϕ graphed below, find

(a) $\lim\limits_{x\to -2^-} \phi(x)$ (b) $\lim\limits_{x\to -2^+} \phi(x)$
(c) $\lim\limits_{x\to -2} \phi(x)$ (d) $\phi(-2)$
(e) $\lim\limits_{x\to -\infty} \phi(x)$ (f) $\lim\limits_{x\to +\infty} \phi(x)$.



6. For the function G graphed below, find

(a) $\lim\limits_{x\to 0^-} G(x)$ (b) $\lim\limits_{x\to 0^+} G(x)$
(c) $\lim\limits_{x\to 0} G(x)$ (d) $G(0)$
(e) $\lim\limits_{x\to -\infty} G(x)$ (f) $\lim\limits_{x\to +\infty} G(x)$.



Limits
(To the tune of "Fins" by Jimmy Buffet)

I am riding on a funky function
Checking it out as x goes to a
I'm coming in from the left side
seeing the y-values along the way

Then I jump over to the right side
And move towards the questionable x
Are both sides approaching the same y-value
That's the question I must answer next

Are they going to the same y-value?
Will both sides of the function meet?

You got a limit from the left
Limit from the right
If they're the same
You got a limit ALRIGHT!!

Whoa Whoa
Whoa Whoa

You got a limit from the left
Limit from the right
If they're the same
You got a limit ALRIGHT!!

If there is an asymptote or break at a
A limit will be hard to find
But don't you discount a hole because
It's scares you cause he's undefined

A limit is a value that you approach
It can be a value that is undefined
So if both sides are going towards the y
A limit we still can find

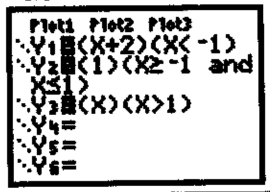
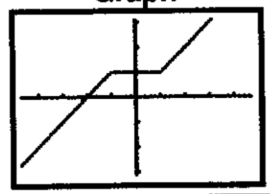
Are they going to the same y-value?
Will both sides of the function meet?

You got a limit from the left
Limit from the right
If they're the same
You got a limit ALRIGHT!!

Pink 2

Original Function	Original Function	Original Function
$y=\frac{x-1}{x+4}$	$y=\frac{x^2-1}{x-1}$	$y=\frac{x+1}{x^2-1}$
Veritcal Asymptotes	**Vertical Asymptotes**	**Vertical Asymptotes**
x = -4	NONE	x = 1
Horizontal/Slant Asymptote	**Horizontal/Slant Asymptote**	**Horizontal/Slant Asymptote**
y = 1	NONE	y = 0
x-intercept	**x-intercept**	**x-intercept**
(1,0)	(-1,0)	NONE
y-intercept	**y-intercept**	**y-intercept**
$\left(0,-\frac{1}{4}\right)$	(0,1)	(0,−1)
Holes	**Holes**	**Holes**
None	x = 1	x = -1

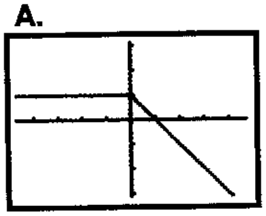
CAN Y=|X| BE WRITTEN AS A PIECE-WISE FUNCTION?

Piece-wise Function	defined on TI-83	Graph
$f(x)=\begin{cases} x+2 & x<-1 \\ 1 & -1\le x\le 1 \\ x & x>1 \end{cases}$		

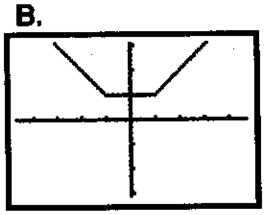
Match each function with a graph.

1) $f(x)=\begin{cases} -x & x<0 \\ x & x\ge 0 \end{cases}$	2) $f(x)=\begin{cases} 2-x & x<2 \\ x-2 & x\ge 2 \end{cases}$	3) $f(x)=\begin{cases} -x & x<2 \\ x-4 & x\ge 2 \end{cases}$	4) $f(x)=\begin{cases} 1 & x<0 \\ 1-x & x\ge 0 \end{cases}$
5) $f(x)=\begin{cases} 1 & x<-1 \\ x^2 & -1\le x\le 1 \\ 1 & x>1 \end{cases}$	6) $f(x)=\begin{cases} x^2-1 & x<-1 \\ 1-x^2 & -1\le x\le 1 \\ x^2-1 & x>1 \end{cases}$	7) $f(x)=\begin{cases} -x & x<-1 \\ 1 & -1\le x\le 1 \\ x & x>1 \end{cases}$	
8) $f(x)=\begin{cases} -x & x<-1 \\ 1 & -1\le x\le 1 \\ 2-x & x>1 \end{cases}$	9) $f(x)=\begin{cases} \dfrac{x^2-1}{x-1} & x\ne 1 \\ 2 & x=1 \end{cases}$	10) $f(x)=\begin{cases} \dfrac{x^2-1}{x-1} & x\ne 1 \\ 1 & x=1 \end{cases}$	

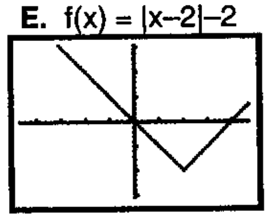
A.



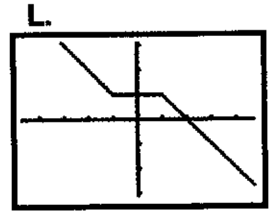
B.



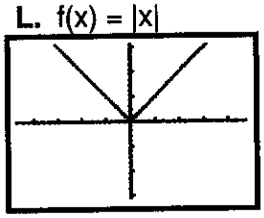
E. $f(x) = |x-2|-2$



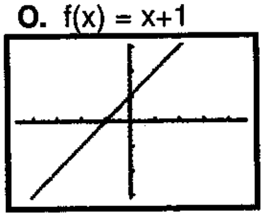
L.



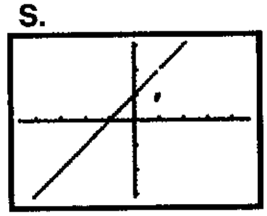
L. $f(x) = |x|$



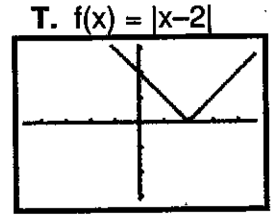
O. $f(x) = x+1$



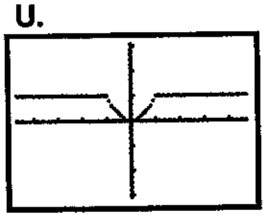
S.



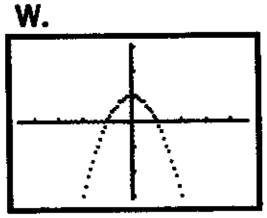
T. $f(x) = |x-2|$



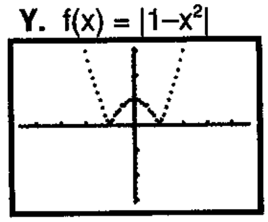
U.



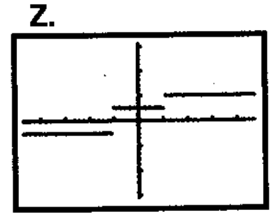
W.



Y. $f(x) = |1-x^2|$



Z.



4	7	10	9	8	5	2	3	1	6

!

92

Piecewise Functions and Continuity

1. a) Graph $y = |x+3| - |$

   b) Write the function as a piecewise function.

2. Graph $y = f(x)$. $f(x) = \begin{cases} 2x-5 & ; x < -1 \\ -3x-2 & ; x \geq -1 \end{cases}$

   a) State the intervals of x where the function is continuous.

   b) State the domain and range for the function

   c) Find $f(2)$

   d) Find $f(-8)$

   e) Find the value of x where $f(x) = -10$.

3. a) Graph $y = f(x)$. $f(x) = \begin{cases} 2x+1 & ; x < -1 \\ 3 & ; x \geq -1 \end{cases}$

   b) State the intervals of x where the function is continuous.

4. Find the value of (a) for which the function is continuous over the interval $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^2+2 & ; x < 0 \\ -x^2+a & ; x \geq 0 \end{cases}$$

5. Find the value of (a) for which the function is continuous over the interval $(-\infty,\infty)$.

$$f(x) = \begin{cases} 3x & ; x<-2 \\ 2x+a & ; x \geq -2 \end{cases}$$

Green 1

Original Function	Original Function	Original Function
$y=\dfrac{x+3}{x-2}$	$y=\dfrac{(2x-2)(x-1)}{(x+3)(x-1)}$	$y=\dfrac{(x-2)(x-3)}{(x-2)(2x+4)}$
Veritcal Asymptotes	**Vertical Asymptotes**	**Vertical Asymptotes**
x = 2	x = -3	x = -2
Horizontal/Slant Asymptote	**Horizontal/Slant Asymptote**	**Horizontal/Slant Asymptote**
y = 1	y = 2	$y=\dfrac{1}{2}$
x-intercept	**x-intercept**	**x-intercept**
(-3,0)	NONE	(3,0)
y-intercept	**y-intercept**	**y-intercept**
$\left(0,-\dfrac{3}{2}\right)$	$\left(0,-\dfrac{2}{3}\right)$	$\left(0,-\dfrac{3}{4}\right)$
Holes	**Holes**	**Holes**
None	x =1	x =2

Orange 4

Original Function $y=\dfrac{x^3+5x^2+6x}{x^2+5x}$	Original Function $y=\dfrac{x^2-7x}{x-5}$	Original Function $y=\dfrac{(-x^2+7x-6)(x+5)}{x^2-2x-35}$
Veritcal Asymptotes x = -5	Vertical Asymptotes x = 5	Vertical Asymptotes x = 7
Horizontal/Slant Asymptote $y=x$	Horizontal/Slant Asymptote $y=x-2$	Horizontal/Slant Asymptote $y=-x$
x-intercept (-3,0), (-2,0)	x-intercept (0,0), (7,0)	x-intercept (1,0),(6,0)
y-intercept NONE	y-intercept (0,0)	y-intercept $\left(0,\frac{6}{7}\right)$
Holes x = 0	Holes NONE	Holes x = -5

Blue 3

Original Function $y=\frac{(x^2+3x-4)(x+4)}{x^2+7x+12}$	Original Function $y=\frac{x^2-3x}{x-4}$	Original Function $y=\frac{(x^2+3x+2)(x-4)}{x^2-7x+12}$
Veritcal Asymptotes x = -3	Vertical Asymptotes x = 4	Vertical Asymptotes x = 3
Horizontal/Slant Asymptote y = x	Horizontal/Slant Asymptote y = x +1	Horizontal/Slant Asymptote y = x + 6
x-intercept (1,0)	x-intercept (0,0), (3,0)	x-intercept (-1,0),(-2,0)
y-intercept $\left(0,-\frac{4}{3}\right)$	y-intercept (0,0)	y-intercept $\left(0,-\frac{2}{3}\right)$
Holes x = -4	Holes NONE	Holes x = 4

Slopes of Curves

1. For $3y^2 - 2x + 4y = 10$, the slope function m for any point on the curve is given by $m = \dfrac{1}{3y+2}$.

a) Is this slope ever zero? Show why or why not.

b) What does this mean about the graph?

c) When would the slope be undefined? Show steps.

d) Describe the tangent line to the curve where the slope is undefined.

e) When would the slope be positive? Describe the curve when the slope is positive.

f) Use parametric equations to graph the curve
[Solve for x and put in x(t). Let y(t) = t]

g) Verify your answers to part (f) by looking at the graph.

2. For $y = \dfrac{4}{3}x^3 - 2x^2 - 8x + 1$, the slope function m for any point on the curve is given by $m = 4x^2 - 4x - 8$.

a) Find the points where the slope would be zero. Show the algebraic steps.

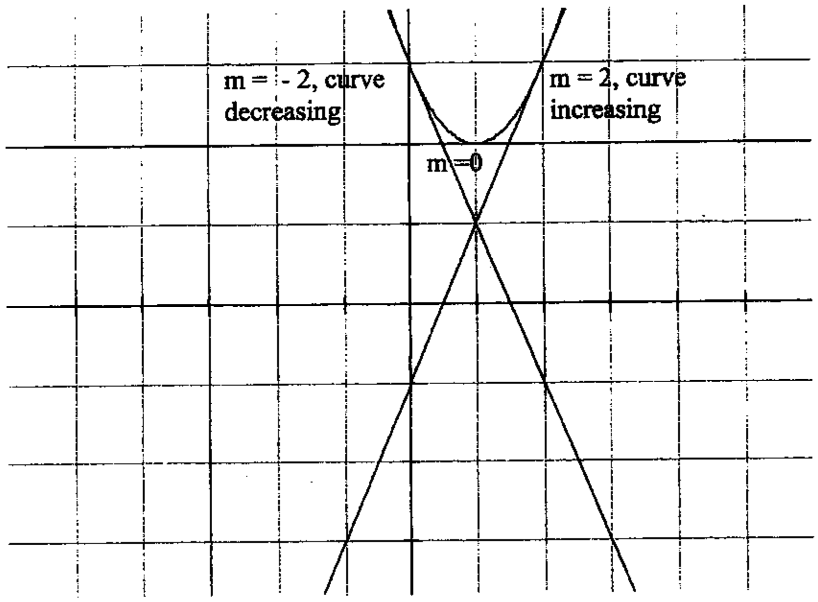
b) Write the equation(s) of the horizontal tangent line(s) to the curve at the point(s).

c) Determine if the curve has a maximum or minimum value at the point(s).

3. Consider the curve $2y^3 + 6x^2y - 12x^2 + 6y = 1$. The slope function m of the curve at all points is given by $m = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$.

a) Find the slope of the curve at the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

b) Is the curve increasing or decreasing at the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$? Explain.



Example:
For $y = 3x^2 - 2x + 4$ the slope function m at any point on the curve is $m = 6x - 2$.

Draw the graph of the parabola and draw the tangent line to the curve at the point (2, 12). Since the slope function is $6x - 2$, at the point (2, 12) the slope of the line tangent would be $6(2) - 2 = 10$ which, as can be seen from the graph, signifies that the curve is increasing at that point.

a) Use the slope function to determine if the curve is increasing at $x = 0$.

b) Write the equation of the line tangent to the curve at $x = -1$.

c) Find algebraically where the slope of the tangent line to the curve would be horizontal. Does a maximum or minimum occur there? [Show testing to see if the slope changes from negative to positive or positive to negative at points on either side of where it is zero]. Since the curve is a parabola what is the name given to this maximum or minimum point?

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AP Calculus AB Name:________________

Zoom In!
A Lesson in Local Linearity

What is slope?

<u>Instructions:</u>

Write down your x value from your index card. X = ______

1. Graph the function given. Use a standard window.
2. Press **Trace** and type in your x value given.
3. Record the coordinate for this x-value of your function. (,) This is your original point.
4. Press **Zoom**, **(2)Zoom In**, **Enter**, and then **Trace** to remain on the graph. Keep repeating this process until your portion of the graph is "linear enough" for you. <u>Don't forget to press trace each time!</u>
5. Once satisfied with your linear portion of the curve, move one pixel to the right or left. Record this coordinate as your second point. (,).
6. Quit to the home screen to calculate the slope using these two points. Your screen should be set up this way:

   (original y – y_1)/ (x – your x value) = record this value as "my slope"

   **Note: If you move from the graph to the home screen, the calculator temporarily stores your x and y value.

   <u>Do not do anything with the graph of classroom slopes or the graph yet.</u>

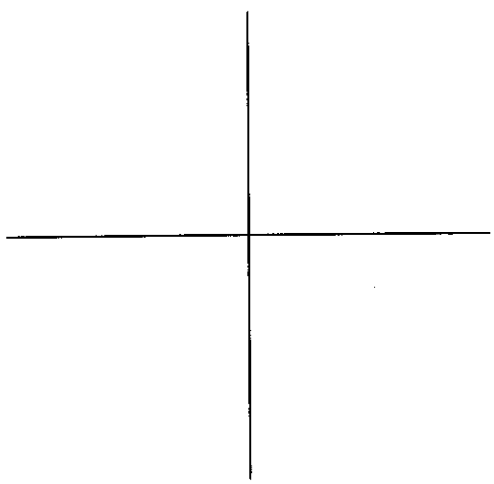
I. $y = x^2$

original point (,)

second point (,)

my slope $m =$ ______________

Graph of Classroom Slopes: y = _____



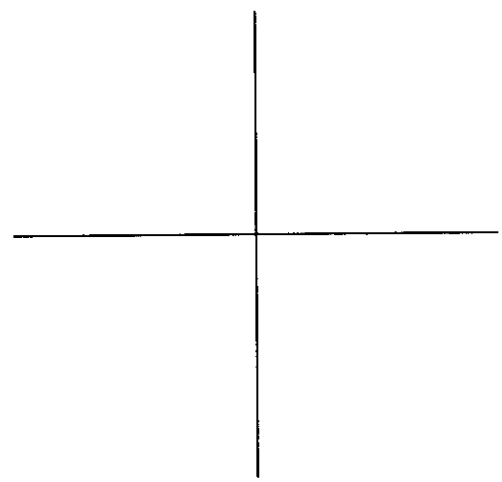
II. $y = x^3$

original point (,)

second point (,)

my slope $m =$ ______________

Graph of Classroom Slopes: y = __________



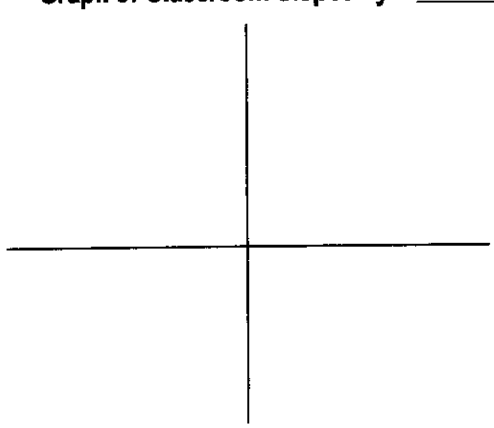
III. $y = x^4$

original point (,)

second point (,)

my slope $m =$ ______________

Graph of Classroom Slopes: y = _____



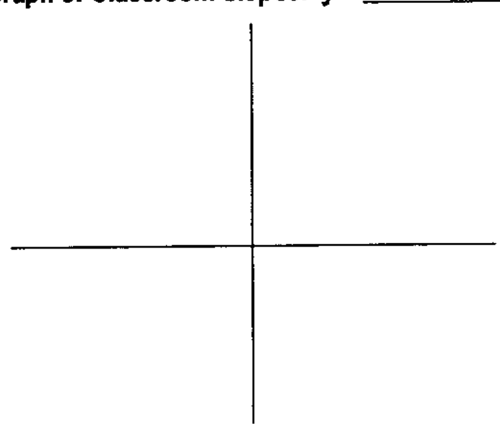
IV. $y = \sin x$

original point (,)

second point (,)

my slope $m =$ ______________

Graph of Classroom Slopes: y = __________



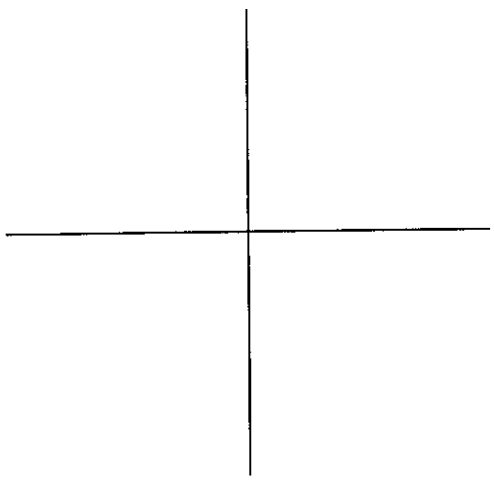
IV. $y = \cos x$

original point (,   )

second point (,   )

my slope $m =$ ____________

Graph of Classroom Slopes: $y =$ _____



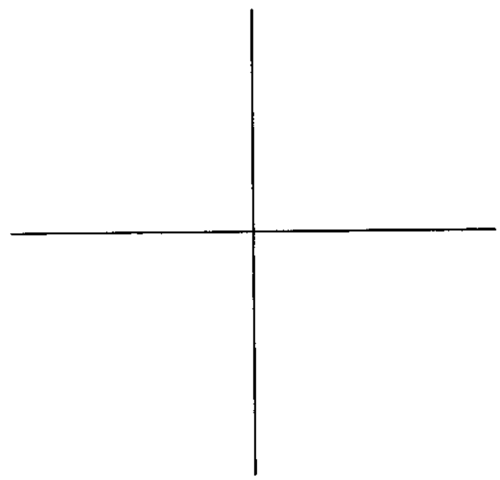
VI. $y = e^x$

original point (,   )

second point (,   )

my slope $m =$ ____________

Graph of Classroom Slopes: $y =$ __________



Characteristics of Functions

1. Complete the following questions for the function below.

   $f(x) = 3x + 2$

x	$f(x)$
-2	
-1	
0	
1	
2	

   a) Copy the table of values to the right on your paper then complete the table of values.

   b) Graph the function on graph paper.

   c) Between which two integers does the graph cross the x-axis? Explain in a complete sentence how to find where the function crosses the x-axis by examining the table of values.

   d) State the y-intercept of the graph. Explain in a complete sentence how to find the y-intercept of the graph by examining the table of values and by looking at the equation.

   e) Is $f(x)$ an increasing or decreasing function? Explain in a complete sentence how you can determine if $f(x)$ is increasing or decreasing by examining the table and by looking at the graph.

2. Complete the following questions for the function below.

   $f(x) = x^2 - 3$

x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	

   a) Copy the table of values to the right on your paper then complete the table of values.

   b) Graph the function on graph paper.

   c) Between which two negative integers does the graph cross the x-axis? Explain in a complete sentence how to find where the function crosses the x-axis by examining the table of values.

   d) State the y-intercept of the graph.

   e) Write the equation for the axis of symmetry for the graph of $f(x)$.

   f) Find the maximum or minimum value for $f(x)$.

   g) Find the values for the domain where $f(x)$ is increasing. Find the values for the domain where $f(x)$ is decreasing.

   h) Find the average rate of change between $x = 0$ and $x = 2$.

3. Complete the following questions for the function below.

   $f(x) = x^3$

x	$f(x)$
-2	
-1	
0	
1	
2	

   a) Copy the table of values to the right on your paper then complete the table of values.

   b) Graph the function on graph paper.

   c) For what values of x is $f(x)$ equal to 0?

   d) Find the y-intercept of the graph.

4. Complete the following questions for the function below.

   $$f(x) = \frac{4}{x}$$

x	$f(x)$
-8	
-4	
-2	
-1	
-0.5	
0	
0.5	
1	
2	
4	
8	

   a) Copy the table of values to the right on your paper then complete the table of values.

   b) Graph the function on graph paper.

   c) Find the value of x where $f(x) = 0$.

   d) Find the y-intercept of the graph.

   e) Does the graph have any symmetry? Explain.

5. Complete the following questions for the function below.

   $f(x) = 2^x$

x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	

   a) Copy the table of values to the right on your paper then complete the table of values.

   b) Graph the function on graph paper.

   c) Find the value x where $f(x) = 0$.

   d) Determine the y-intercept of the graph.

   e) Is $f(x)$ an increasing or decreasing function?

   f) Find the average rate of change for the following intervals of x.

   $0 \le x \le 1$, $1 \le x \le 2$, $2 \le x \le 3$.

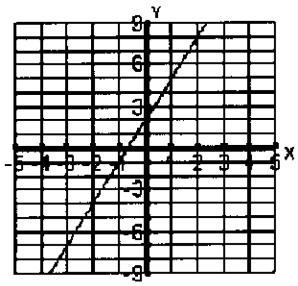
   As the value of x increases, what do you notice about the average rate of change?

Characteristics of Functions

Answers:

1. a)

x	$f(x)$
-2	-4
-1	-1
0	2
1	5
2	8

b) 

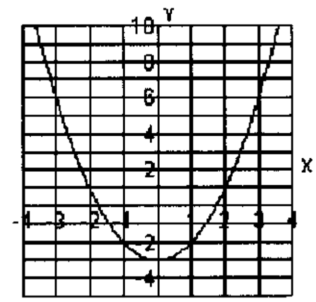
c) $-1 < x < 0$; The graph crosses the x-axis when the value of y changes from negative to positive.

d) 2; The graph crosses the y-axis when the value of x is 0.

e) $f(x)$ is increasing; On the table as the values of x increase the values of $f(x)$ increase and the graph slants upward to the right.

2) a)

x	$f(x)$
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

b) 

c) $-2 < x < -1$; The function crosses the x-axis when the value of y changes from positive to negative, or from negative to positive.

d) -3

e) x = 0

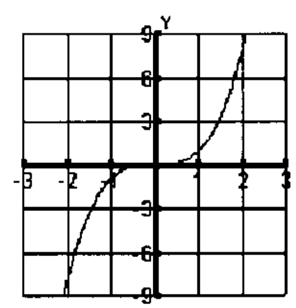
f) Minimum value is -3; no maximum.

g) increasing $x \geq 0$; decreasing $x \leq 0$

h) 2

3. a)

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8

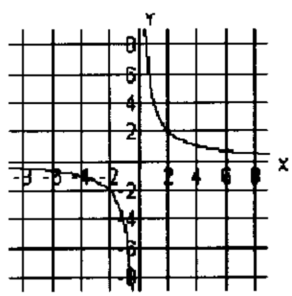
b) 

c) $x = 0$

d) $y = 0$

4. a)

x	$f(x)$
-8	-0.5
-4	-1
-2	-2
-1	-4
-0.5	-8
0	undefined
0.5	8
1	4
2	2
4	1
8	0.5

b) 

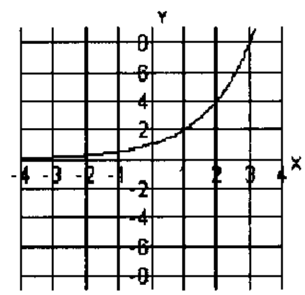
c) undefined or never

d) none

e) $y = x$ and $y = -x$

5. a)

x	$f(x)$
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8

b) 

c) undefined or never

d) $y = 1$

e) increasing

f) 1; 2; 4
The average rate of change is increasing.

SHIPS IN THE FOG

Situation:

Two ships are sailing in the fog and are being monitored at ten-second intervals by Ensign Pulver. As they enter his radar screen at 6:00 A.M., the Minnow is 1000 meters above the lower left corner of the screen, which will henceforth be referred to as the "Origin". The good ship Lollipop is 8000 meters to the right of the origin.

After ten seconds, the Minnow is 40 meters east and 10 meters north of its original position. The Lollipop is 30 meters west and 20 meters north of its original position.

You may assume that the two ships maintain their courses and speeds throughout the problem.

Problem:

A. Determine whether the two ships collide.

1. If they do collide, determine the time and the position of the collision with respect to the origin.

2. If they do not collide, determine the minimum distance between the ships and when they are closest.

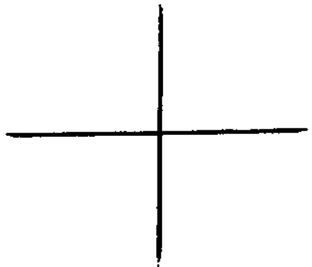
B. Turn in the work which confirms your conclusion. This may either be an analytical explanation or a description of how technology was used to solve the problem.

C. Prepare a graph to illustrate the situation. The graph should be labeled in such a way as to be understandable to a casual observer. The names of your group members should be proudly displayed on your graph.

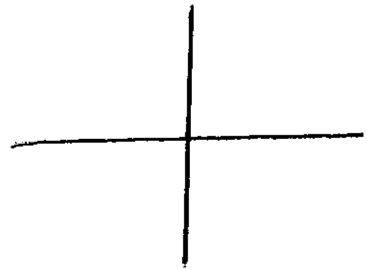
Graphing Polar Equations

On problems 1 - 4, put your graphing calculator in **POLAR** mode and **RADIAN** mode. Graph the following equations on your calculator, sketch the graphs, and answer the questions.

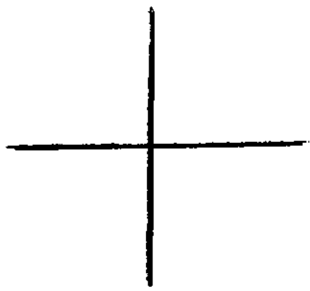
1. $r = 2\cos\theta$



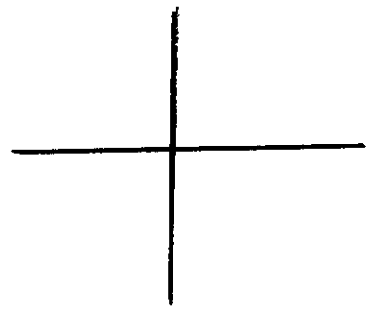
$r = -3\cos\theta$



$r = 2\sin\theta$

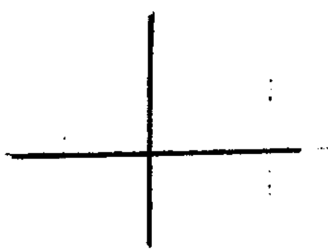


$r = -3\sin\theta$

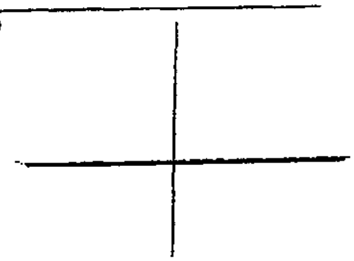


What do you notice about these graphs?

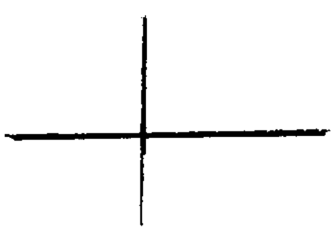
2. $r = 2 + 2\cos\theta$



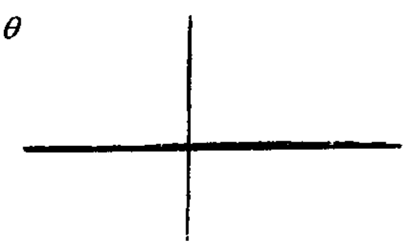
$r = 1 + 2\cos\theta$



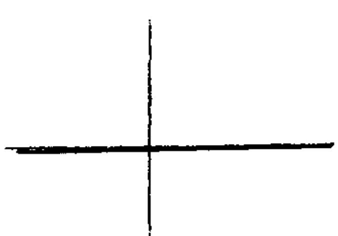
$r = 2 + \cos\theta$



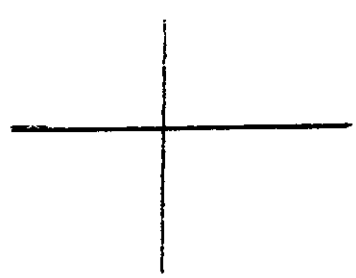
$r = 2 + 2\sin\theta$



$r = 1 + 2\sin\theta$

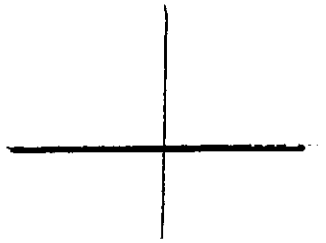


$r = 2 + \sin\theta$

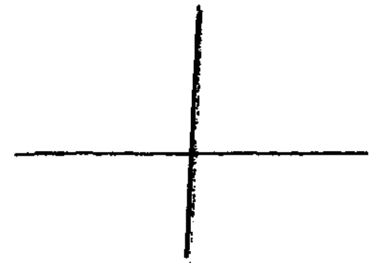


Which graphs go through the pole (origin)?
Which ones do not go through the pole?
Which ones have an inner loop?
What causes these things to happen? (Hint: Go to **FORMAT** and set your calculator to show the polar graphing coordinates when you trace.)

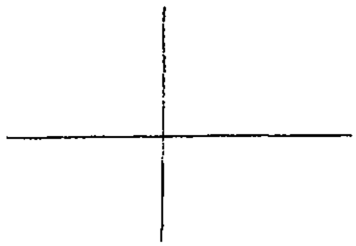
3. $r = 3\cos 3\theta$



$r = 2\cos 5\theta$



$r = 3\sin 3\theta$



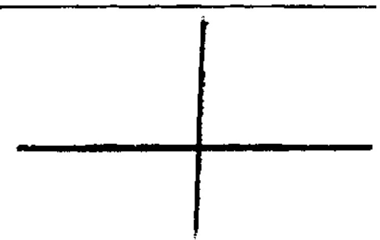
$r = 2\sin 5\theta$

What do you notice about these graphs?

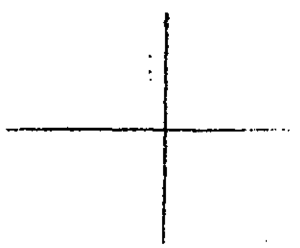
---

4. $r = 2\cos 2\theta$

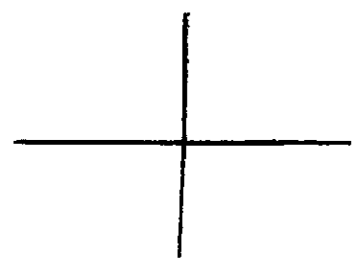
$r = 3\cos 4\theta$



$r = 2\sin 2\theta$



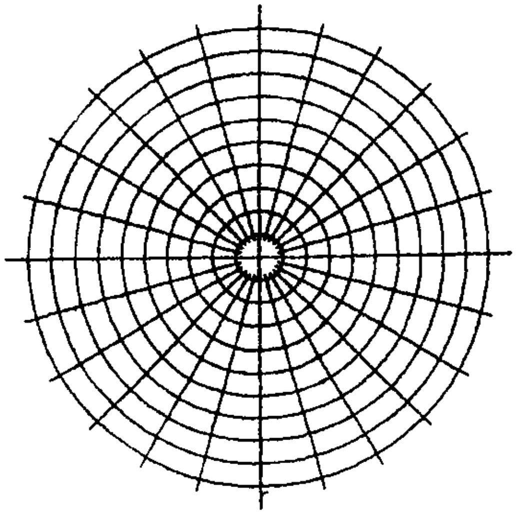
$r = 3\sin 4\theta$

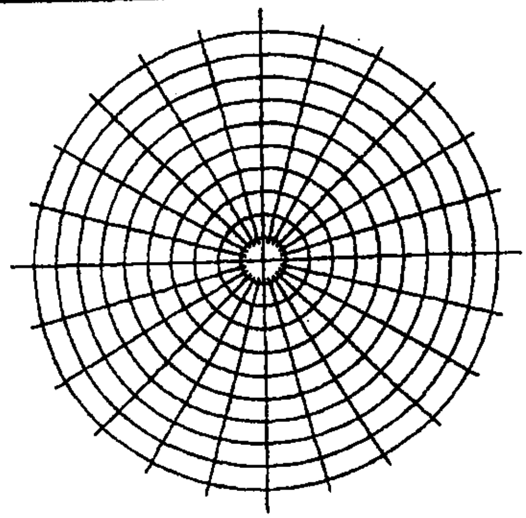


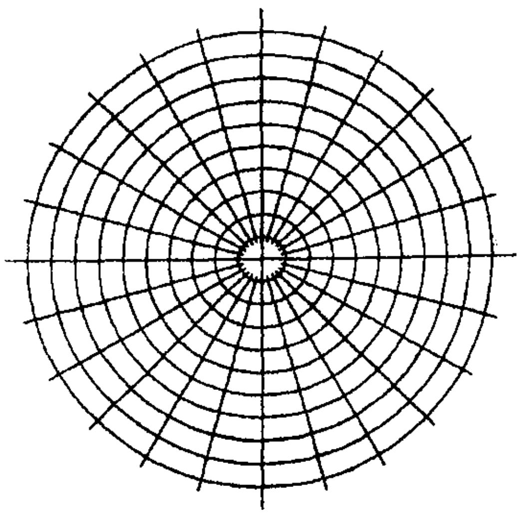
What do you notice about these graphs?

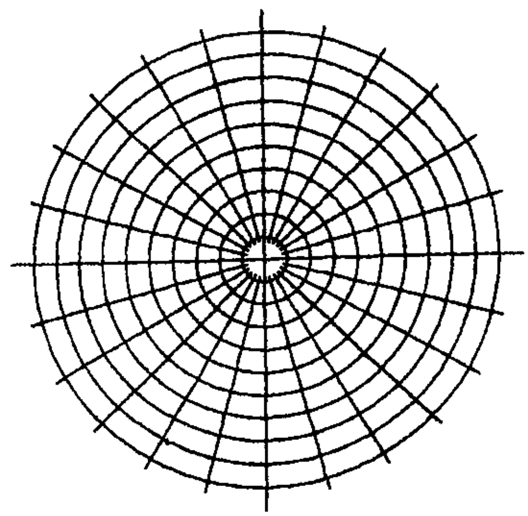
Name ____________________ Group ____________________ Date ____________

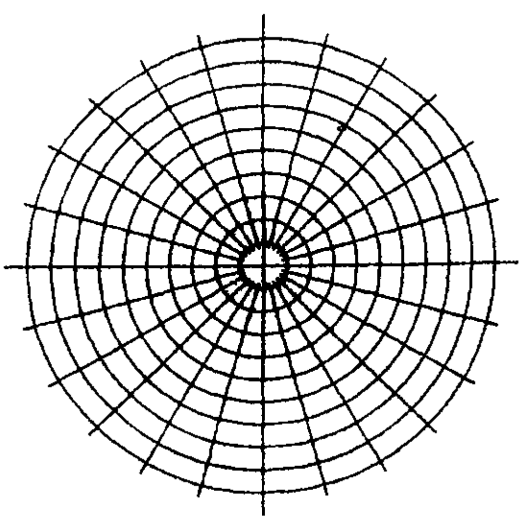
Polar Coordinate Paper

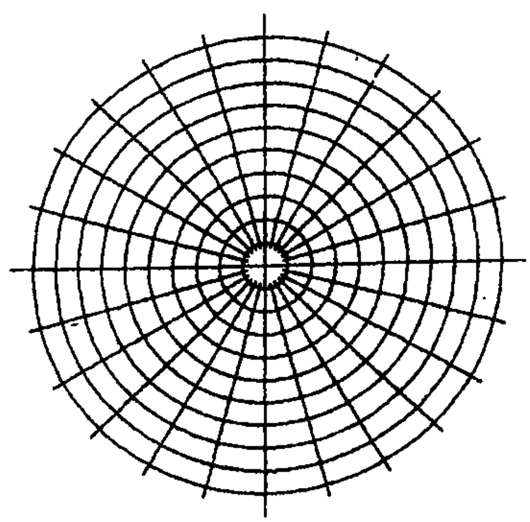






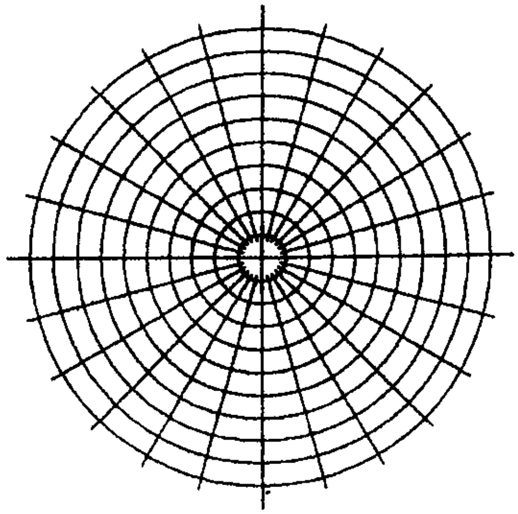


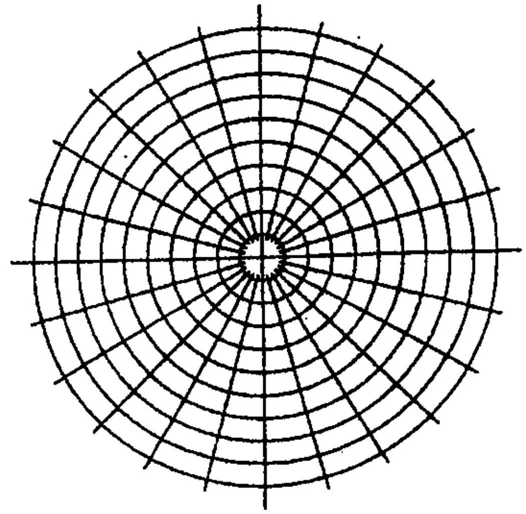


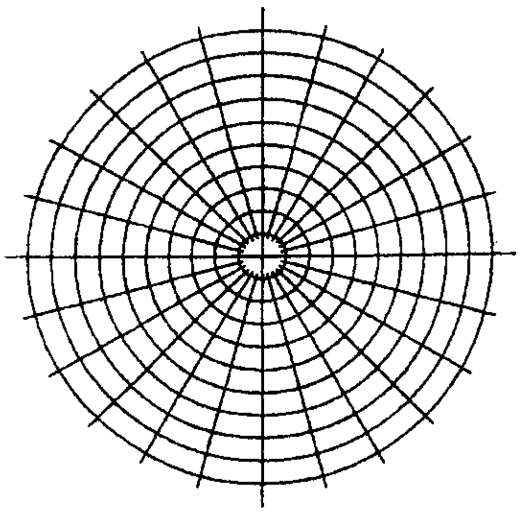


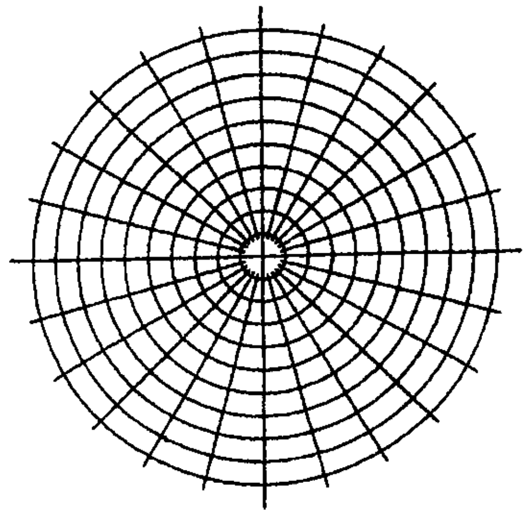
Name ______________________ Group ________________ Date __________

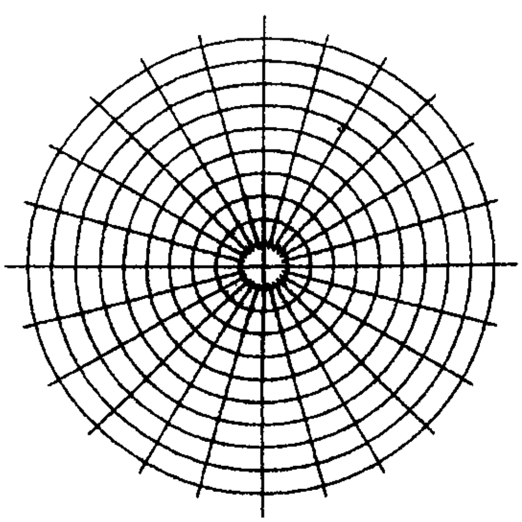
Polar Coordinate Paper

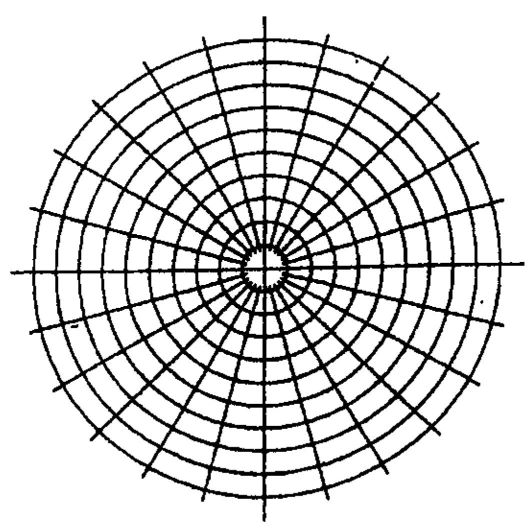






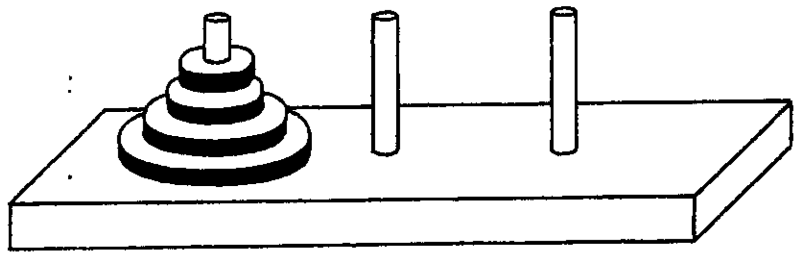






TOWER OF HANOI

The object of this ancient puzzle is to transfer the tower of discs to either of the two vacant pegs in the fewest possible moves. You may only move one disc at a time. You may not place a disc on one that is smaller.



In the table to the right, *n* represents the number of discs in the tower. *M* represents the fewest number of moves it takes to transfer those discs to the vacant pegs.

What is the fewest number of moves with four discs?

Complete the table at the right through seven discs.

Find a pattern which would give you the solution for 64 discs.

How did you find the pattern?

What is the formula for the fewest number of moves needed to transfer *n* number of discs?

Find a pattern for the number of moves each disc makes. Consider the smallest disc number one.

n	*M*
1	______
2	______
3	______
4	______
5	______
6	______
7	______
⋮	
64	______
⋮	
n	______

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Fibonacci numbers

1
                    1
2
                    3
Fibo
                    nacci
numbers are we.   numbers are we.

5
                    8
13
                    21
We go on and on.   We go on and on.

Always
                    adding the two last of us
gives the next.   gives the next.

301

Spirals	1
pine cones	1
hexagons	2
pineapples	3
shells	5
leaves	8
flowers	13
fruits	21
phyllotaxis	34
golden ratio	55

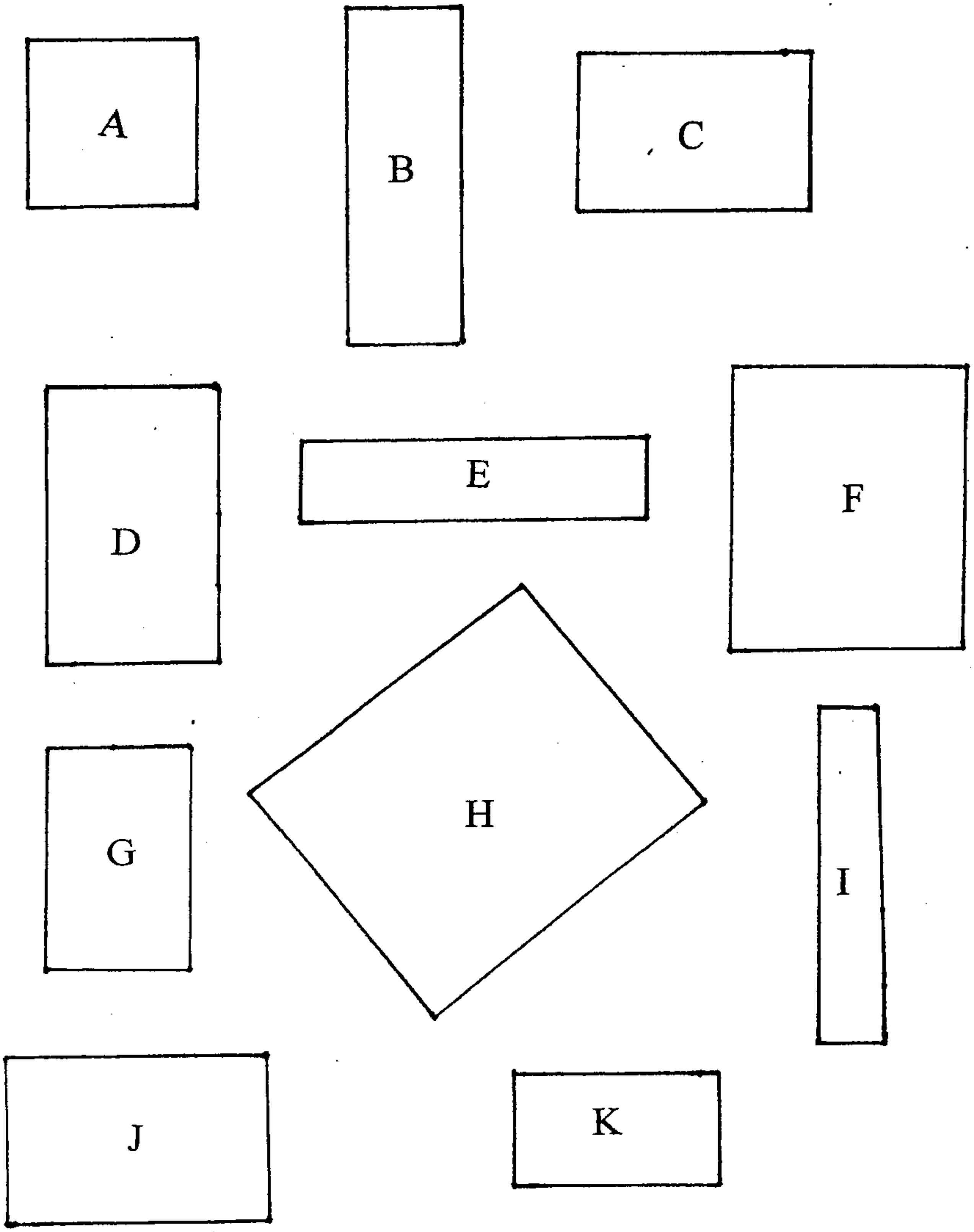
We continually appear
in so many places
and things.
Nature
Pascal's triangle
pentagrams
infinite series
the golden
mean.

Fibo
nacci
numbers are we.   numbers are we.

FIBONACCI

1,1,2,3,5,8,13....

Which Rectangles Appeal to You?



Choices: First ______ Second ______ Third ______

WRITE THE FOLLOWING RATIOS IN DECIMAL FORM USING VALUES FROM YOUR TABLE

RATIO #1: express $\frac{Y}{X}$ = __________ in decimal form (to 4 places)

RATIO #2: express $\frac{Y+X}{Y}$ = __________ in decimal form (to 4 places)

These ratios should be approximately equal to each other for your belly button to be proportionately placed on your body. The closer the ratio is to 1.618 the more perfect you are.

1. If RATIO #1 is larger than 1.618 which portion of your body is too long: torso or legs ?
2. If RATIO #1 is smaller than 1.618 which portion of your body is too long: torso or legs?

NOW TO CALCULATE YOUR PERFECTION. USE RATIO #1

STEP 1: absolute value of (RATIO #1 – 1.618) = $\boxed{}$

STEP 2: $\boxed{}$ divided by 1.618 = $\boxed{}$ (4 places)

STEP 3: 1 – $\boxed{}$ = $\boxed{}$

STEP 4 $\boxed{}$ times 100 = $\boxed{\%}$ To the nearest tenth

THIS NUMBER REPRESENTS HOW PERFECT YOUR BELLY BUTTON IS !!!

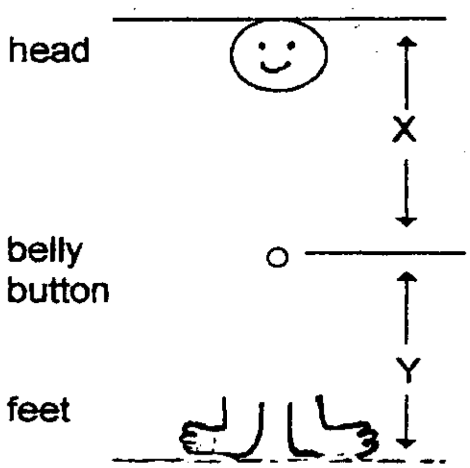
HOW PERFECT IS YOUR BELLY BUTTON?

A golden section is a line segment that has been divided into two parts in such a way that the ratio of the longer part "Y" to the shorter part "X" is equal to the ratio of the entire segment "Y + X" to the longer part "Y". This is known as the *GOLDEN RATIO.*

$$\frac{Y}{X} = \frac{Y+X}{Y}$$

When you cross multiply you get a quadratic which when solved for "Y" in terms of "X" this ratio should have a value of $\frac{1 \pm \sqrt{5}}{2}$ which is approximately equal to 1.618. Historically it has been thought that a form is most pleasing when its parts divide it in golden sections.

According to the ancient Greeks, one's body is proportional if certain distances satisfy the *GOLDEN RATIO.* You are going to discover just how perfectly proportioned you are.

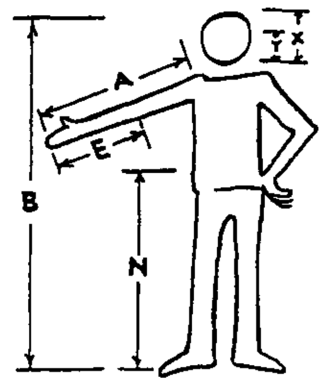


You will need a measuring tape or ruler , calculator & this worksheet.

Begin by making the following measurements and recording them in the chart below.

DESCRIPTION	VALUE
X = distance from your belly button to your head	X=
Y = distance from your belly button to your feet	Y=
H = your height (should equal X+Y)	H =

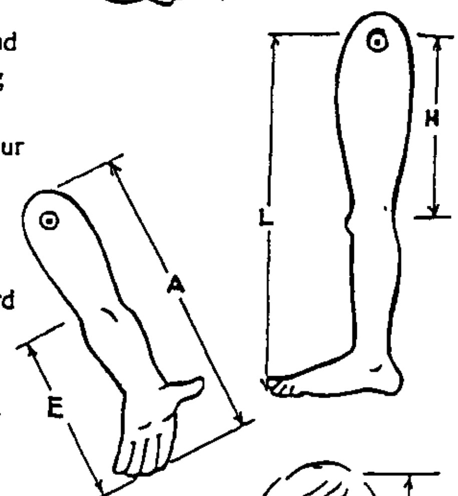
American researcher, Jay Hambridge, established that indeed the Golden Ratio can be found not only in Greek temples and sculpture, but also in the proportions of the human skeleton. The ratio of the total height to the height of the navel is a close approximation to the Golden Ratio. Other writers and researchers have claimed that ratios of many other parts of the human body are also in the Golden Ratio. In other words, we probably all have some proportions close to the Greek ideal somewhere in our bodies.



Let's see if the Golden Ratio is somewhere in each of us. Form groups of four or five. Make a table like the one on the bottom of this page. Include the name of each person in your group.

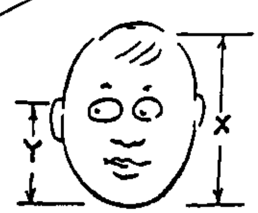
Step 1: Measure the height (B) and the navel height (N) of each member of your group. Calculate the ratios B/N. Record them in your table.

Step 2: Measure the length (F) of an index finger and the distance (K) from the finger tip to the big knuckle of each member of your group. Calculate the ratios F/K. Record them in your table.



Step 3: Measure the length (L) of a leg and the distance (H) from the hip to the kneecap of everyone in your group. Calculate and record the ratios L/H.

Step 4: Measure the length (A) of an arm and the distance (E) from the finger tips to the elbow of everyone in your group. Calculate and record the ratios A/E.



Step 5: Select another pair of lengths (X and Y) on the body that you suspect may be in the golden ratio. Measure these lengths. Calculate the ratios (large to small) and record them.

Person's name	B/N	F/K	L/H	A/E	X/Y
1. —?—	—?—	—?—	—?—	—?—	—?—
2. —?—	—?—	—?—	—?—	—?—	—?—
⋮	⋮	⋮	⋮	⋮	⋮

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IN SEARCH OF THE GOLDEN RATIO

a_n	$r = a_n / a_{n-1}$
1	
1	1
2	2
3	1.5
5	1.666666667
8	1.6
13	1.625
21	1.615384615
34	1.619047619
55	1.617647059
89	1.618181818
144	1.617977528
233	1.618055556
377	1.618025751
• • •	• • •
Infinity	Golden Ratio

Pascal's Triangle

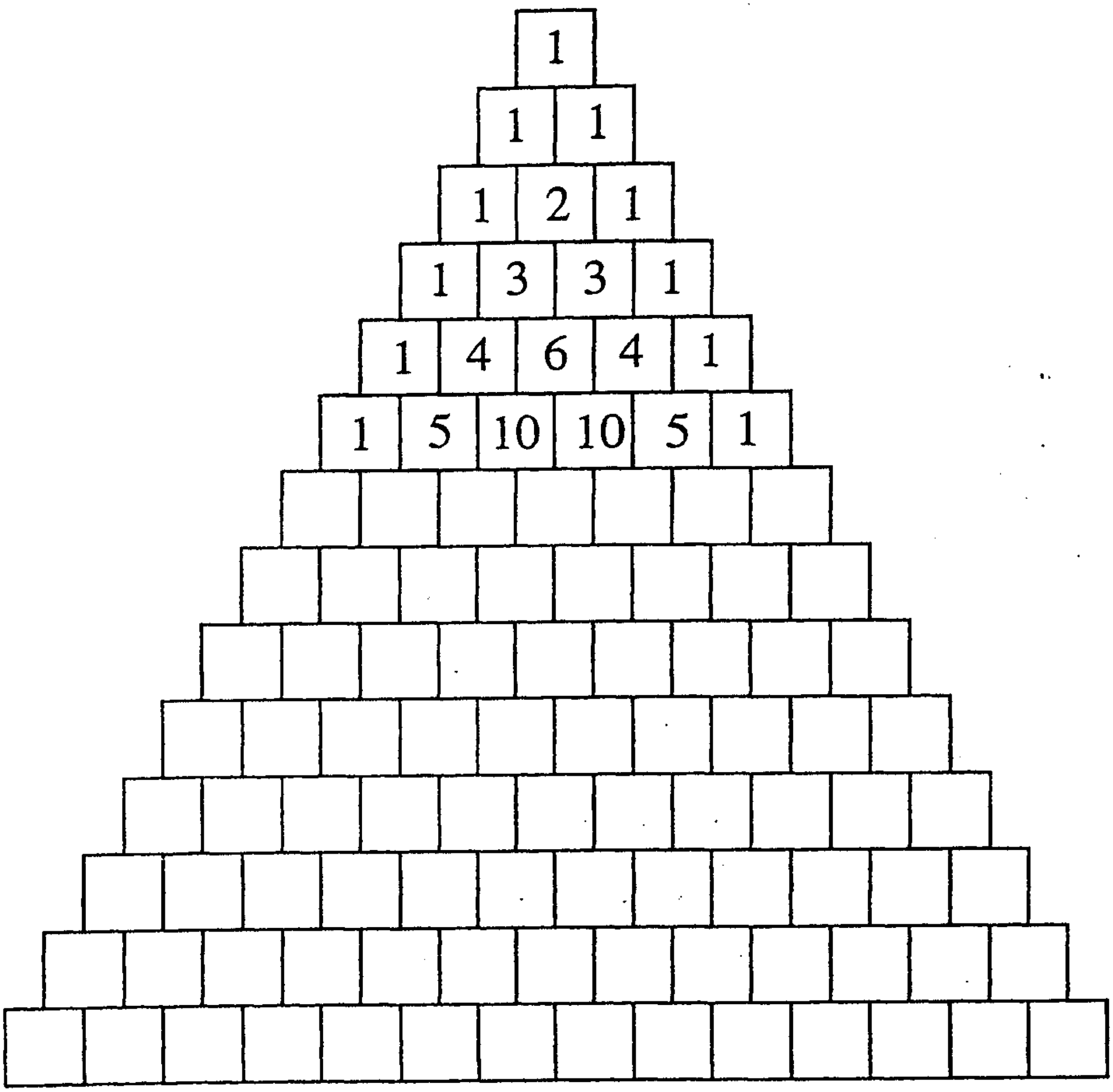

CHART 1

PASCAL'S TRIANGLE

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11 440 12 870 11 440 8008 4368 1820 560 120 16 1

SPECIAL SEQUENCES

(1) ARITHMETIC

$$a_1, a_1+d, a_1+2d, \ldots, a_1+(n-1)d, \ldots$$

Explicit : $\{a_n = a_1 + (n-1)d\}$

Recursive: $\left\{\begin{aligned} &a_1 = a_1 \\ &a_n = a_{n-1} + d \ \ (n \geq 2) \end{aligned}\right\}$

(2) GEOMETRIC

$$a_1, a_1 r, a_1 r^2, \ldots, a_1 r^{n-1}, \ldots$$

Explicit : $\{a_n = a_1 r^{n-1}\}$

Recursive : $\left\{\begin{aligned} &a_1 = a_1 \\ &a_n = a_{n-1} r \ (n \geq 2) \end{aligned}\right\}$

(3) FIBONACCI

$$a_1, a_2, a_2 + a_1, a_3 + a_2, \ldots, a_{n+1} + a_n, \ldots$$

Recursive : $\left\{\begin{aligned} &a_1 = a_1 \\ &a_2 = a_2 \\ &a_n = a_{n-1} + a_{n-2} \ (n \geq 3) \end{aligned}\right\}$

12 – 1: Arithmetic Sequences and Series

- A **sequence** is a set of numbers in a specific order. The **1st term** is a_1, the **2nd term** is a_2...and the **n^{th} term** is a_n. The difference between successive (or consecutive) terms of an arithmetic sequence is called the **common difference**, denoted as d. A sequence is of the form:

$$a_1,\ a_1+d,\ a_1+2d,\ \ldots$$

The common difference can be found by subtracting any term from its succeeding term.

- To find the next term in an arithmetic sequence, find the common difference by subtracting two consecutive terms, and add the difference to the last term that you know.

- **EXAMPLES:**

1. Find the next three terms in the sequence: $-12,\ -1,\ 10,\ \ldots$

$d =$ ________

____

2. Find the next three terms in the sequence: $r-4,\ r-1,\ r+2,\ \ldots$

$d =$ ________

____

- A **recursive formula** is one in which each succeeding term is formulated from one or more previous terms. The n^{th} term of an arithmetic sequence can be found by using such a formula where the 1st term and common difference are known.

The n^{th} term of an arithmetic sequence is given by:	
$a_n = a_1 + (n-1)d$	
a_1 is the 1st term	n is the number of terms
a_n is the n^{th} term	d is the common difference

- **EXAMPLES:**

3. Find the 41st term in the sequence: $11,\ 4,\ -3,\ \ldots$

$d =$ ________

$a_{41} =$ ________

4. Find the 1st term in the sequence for which $a_{44} = -229$ and $d = 8$

$n =$ ________

$a_1 =$ ________

- The terms between any two non-consecutive terms of an arithmetic sequence are called **arithmetic means**. Find the common difference, then insert the terms into the sequence.

- **EXAMPLE:**

5. Form an arithmetic sequence that has six arithmetic means between -12 and 23.

$a_1 =$ ________ $n =$ ________ $a_8 =$ ________ $d =$ ________

-12, ____, ____, ____, ____, ____, ____, 23

- An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence.

Arithmetic Sequence: $a_1, a_2, a_3, \dots, a_n$ ***Arithmetic Series:*** $a_1 + a_2 + a_3 + \dots + a_n$

The sum of the first n terms of an arithmetic series is:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

n is the number of terms a_1 is the 1^{st} term

S_n is the sum of n terms a_n is the n^{th} term

- **EXAMPLES:**

6. Find the sum of the first 32 terms in the series $-12 - 6 - 0 - \dots$

You need the 1^{st} and last terms of the series and the number of terms. You have a_1 and n, but not a_n.

What formula do you have to find a_n?

$a_1 =$ ________

$n =$ ________

$d =$ ________

$a_{32} =$ ________

$S_{32} =$ ____________

Another formula you can use if you do not know the last term is:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

7. Lauren starts a college savings account for her daughter on her sixth birthday. She plans to deposit \$25 the first month and then increase the deposit by \$5 each month. How much will she deposit in twelve years?

$n =$ ________

$a_1 =$ ________

$d =$ ________

$a_{144} =$ ________

$S_{144} =$ ________

12 – 2: Geometric Sequences and Series

- A **geometric sequence** differs from an arithmetic sequence in that successive (or consecutive) terms are found by using a **common ratio**, denoted as r, rather than a common difference. It has the form: $a_1, a_1r, a_1r^2, \ldots$

  The common ratio can be found by dividing any term by the preceding term.

- **EXAMPLE:**

1. Find the next three terms in the geometric sequence: 18, 54, 162, …

$r =$ ______

____

- A geometric sequence can also be found by a recursive formula where the 1st term and common ratio are known:

The n^{th} term of an geometric sequence is given by:
$a_n = a_1 \cdot r^{n-1}$
a_1 is the 1st term n is the number of terms
a_n is the n^{th} term r is the common ratio

- **EXAMPLE:**

2. Find the 12th term in the sequence: $1, \frac{1}{2}, \frac{1}{4}, \ldots$

$r =$ ______

$a_{12} =$ ______

- The terms between any two non-consecutive terms of an geometric sequence are called **geometric means.** Find the common ratio, then insert the terms into the sequence.

- **EXAMPLE:**

3. Form a sequence that has two geometric means between 128 and 54.

$a_1 =$ ______ $n =$ ______ $a_4 =$ ______ $r =$ ______

128, ____, ____, 54

The sum of the first n terms of a geometric series is:
$S_n = \dfrac{a_1 - a_1r^n}{1-r}, \quad r \neq 1$

- **EXAMPLE:**

4. Find the sum of the first six terms of the series: $2 - 6 + 18 - 54 + \ldots$

$a_1 =$ ______

$n =$ ______

$r =$ ______

$S_6 =$ ______

Infinite Summing

Objective:
To see that an infinite series can have a finite sum and to develop the formula for sum of an infinite series.

Connections to Previous Learning:
Arithmetic operations, sequences

Connections to TEKS:
111.35 c4 ABCD

Connections to AP:
The calculus topic of series

Time Frame:
90 minutes

Materials:
Poster board, scissors, rectangular cookies (Sugar Wafers work the best)

Teacher Notes:
Any food will work if it is in the shape of a rectangle.

The purpose of this assignment is to physically represent the concept of summing infinitely. Dividing a cookie in half as many times as possible, then adding the parts back together will give a physical representation of the series

$\frac{\text{ſ}}{2}+\frac{\text{ſ}}{4}+\frac{\text{ſ}}{8}+\frac{\text{ſ}}{16}+\ldots = 1$ cookie.

The summation formula will be developed in this lesson. In addition, an extension to series concepts important in BC Calculus will be introduced.

For example, $2 + 2(x-1) + 2(x-1)^2 + 2(x-1)^3 \ldots$ may be added together using the formula

$S = \frac{a}{1-r}$ if $|r| < 1$ and in this case $|x-1| < 1$

(in other words r is the ration of two sequential terms and a is the first term).

If $a = 2$, $r = 1 - x$, then the sum of this series is $\frac{2}{1-(x-1)} = \frac{2}{2-x}$. The words converge and diverge will be used.

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Activity 1:

Cut a piece of poster board in half, keep half.

Cut that in half, keep half

Cut that in half, keep half

Cut that in half, keep half.

Cut that in half, keep half.

How long could we continue this activity? (infinitely)

If we took all pieces, not just the half we kept, put them all back together, what would we have? (one piece of poster board)

We have an infinite number of pieces which together make a finite piece. How can that happen???? (the process is infinite not the material)

This will only work when we are taking fractional ($|r| < 1$) pieces of our paper.

To find the sum of the infinite pieces, follow the process:

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots + a_n$$

$$\frac{1}{2}S = \quad \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \ldots + \frac{1}{2}a_n$$

Subtract:

$S - \frac{1}{2} S = \frac{1}{2} - \frac{1}{2}(a_n)$

$\frac{1}{2} S = \frac{1}{2}\ (1 - a_n)$

$S = 1 - a_n$

As we continue cutting, what does a_n become? (the smallest piece will be almost nothing)

Therefore, the sum of all the pieces will equal ______. (1)

Explain what this means. (That no matter how small the pieces get, if you put them all back together like a puzzle, it would create the original piece of poster board.)

Generalize:

Use the process above to generalize a formula for the sum of an infinite geometric series if $|r| < 1$.

$S = a + ar + ar^2 + ar^3 + ar^4 \ldots .ar^{n-1}$

$rS = \quad ar + ar^2 + ar^3 + ar^4 + \ldots ar^{n-1} + ar^n$

Subtract:

$S - rS = a - ar^n$

$S(1 - r) = a(1 - r^n)$

$S = \dfrac{a(1-r^n)}{1-r}$ (As *n* approaches infinity, the fraction *r* to an infinite power approaches 0)

$$S_\infty = \frac{a}{1-r}$$

Activity 2:

Divide 31 cookies among a class of 16.

Give each person a cookie. There will be 15 left.

Divide the 15 remaining cookies in half and give each person a ½. There will be 30 halves to distribute, give away 16 and 14 halves left.

Divide the 14 remaining halves in half again and give each person a ¼. There will be 28 quarters to distribute, give away 16 and 12 quarters will be left.

Divide the 12 remaining quarters of a cookie in half again and give each person an eighth. There will be 24 eighths to distribute, give away 16 and 8 eighths will be left.

How long can this process continue? (theoretically to infinity)

Define iteration. (a repeated process)

When finished, approximately how many cookies have been distributed? (31 cookies)

Will each person theoretically have an infinite number of pieces of cookies to eat? (yes)

Is there a limit to the amount of cookies each person will have to eat? (yes)

What is it? (They will have an infinite number of pieces but a finite amount of $1\frac{15}{16}$ cookies.

Infinite Summing

1. Use the formula for the sum of an infinite series to sum the following series.

   a) $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\frac{1}{54}+\ldots$

   b) $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\ldots$

   c) $4+2\frac{2}{3}+1\frac{7}{9}+\ldots$

   d) $x+x^2+x^3+\ldots if\ |x|<1$

2. Find two different infinite series whose sum is 8.

3. Write an infinite series whose sum is $\frac{3}{1-2x}$.

4. Write an infinite series whose sum is $\frac{3}{2-2x}$.

5. Using the series A and B,

   a) Find A.

   b) Find B.

   c) Write the series A + B.

   d) Find the sum of the series A + B. Does it equal the sum of A plus the sum of B?

   $$A=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\ldots$$
   $$B=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}\ldots$$

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6. Use your calculator to find partial sums of the following infinite series called the harmonic series.

$1+\frac{1}{2}=$

$1+\frac{1}{2}+\frac{1}{3}=$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\ldots\ldots\ldots$

a) Do the terms of this series converge? (Define converge and diverge)

b) Does the sum appear to converge?

c) Can you explain why or why not?

7. $\sum_{0}^{\infty} 3(.2)^x =$

8. $\sum_{0}^{\infty}\left(\frac{1}{2}\right)^X - \left(\frac{1}{3}\right)^X =$

9. Write $\frac{1}{x}$ as an infinite series if $|1-x|<1$.

Infinite Summing

Answers:

1. a. ¾ b. 3 c. 12 d. $\dfrac{x}{1-x}$

2. $8 = \dfrac{1}{\frac{1}{8}} = \dfrac{1}{1-\frac{7}{8}}$ Let $a = 1$ and $r = 7/8$ to create one possible answer.

   $1 + \dfrac{7}{8} + \left(\dfrac{7}{8}\right)^2 + \left(\dfrac{7}{8}\right)^3 \ldots.$

   Consider other possibilities. $8 = \dfrac{2}{\frac{2}{8}} = \dfrac{2}{1-\frac{6}{8}} = \dfrac{2}{1-\frac{3}{4}}$ Let $a = 2$ and $r = ¾$

   $2 + \dfrac{3}{4} + \left(\dfrac{3}{4}\right)^2 + \left(\dfrac{3}{4}\right)^3 \ldots$

3. $\dfrac{3}{2-2x} = \dfrac{3}{2(1-x)} = \dfrac{1}{2} \cdot \dfrac{3}{(1-x)} = ½\,(3 + 3x + 3x^2 + 3x^3 + \ldots)$

4. Method 1 Sum of A + Sum of B = 1 + 1/3 = 4/3

   Method 2 Add A + B to get a new series $1 + \dfrac{2}{8} + \dfrac{2}{32} \ldots = \dfrac{1}{1-\frac{1}{4}} = 4/3$

5.

   a) 1

   b) 1.5

   c) 1.8333

   d) 2.08333, 2.28333, 2.45, . . .

6. The harmonic series diverges since the partial sums increase. The sequence of partial sums must decrease to zero if the original series can be summed. While each successive term is smaller, it is not getting smaller fast enough. Consider the series $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2\ldots$ The partial sums are decreasing because the terms are going to zero faster than the terms in the harmonic series. Rate of growth is a very important concept in series.

7. 15/4

8. 1/2bb

9. $\frac{1}{x}=\frac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+(1-x)^3\ldots$ if $|1-x|<1$.

   To solve this inequality, it is also true that $|x-1|<1$. so -1 $< x-1 <$ 1

   And $0<x<2$.

   This means as long as x is between 0 and 2, the series will have a sum.

Notes to the Teacher

Materials

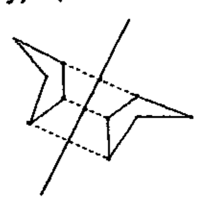
One copy of Blackline Master B1 (3 pages) for each student

(Optional) Students may use the geometric dictionary for this activity if they have passed a qualifying vocabulary quiz. Collect the dictionary with the activity, so that each student is responsible for his/her own dictionary and dictionaries are not shared from class to class and from student to student.

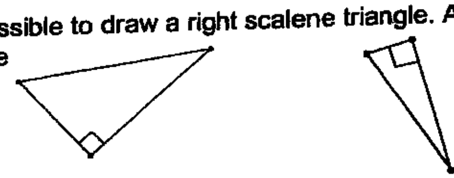
A1 What Do You Know?

Students review concepts taught in previous units. Distribute Blackline Master B1. Students work independently.

Answers:
1. c; 2 h; 3 f; 4 b; 6 g; 9 d; 10 e; 11 a.
Unmatched words (answers may vary):
5. slope: Change in y/change in x; rise/run; $\frac{y_2 - y_1}{x_2 - x_1}$;
7. diagonal: a line segment connecting two non-consecutive vertices;
8. regular octagon: equiangular, equilateral 8-sided polygon.
12. C
13. B
14. $(x, y) \to (x - 5, y + 2)$;
15.



16. It is possible to draw a right scalene triangle. Answers may vary. Some examples:



17. The image coordinates will be at $C'(2, 3)$, $D'(1, -1)$, $E'(4, 0)$.

Materials

One copy of Blackline Master B4 for each student
Three sheets of patty paper for each student
Compasses (can be shared)
Ruler for each student
One blank transparency sheet
Poster paper
Colored markers
Glue or tape

A4 Equilateral Triangle Properties

Ask students to define or give an example of symmetry. Ask the class what is meant by a line of symmetry. Refer to the reflection activity (Unit 2, Lesson 3). Sketch a triangle, a reflection line and its image on the board. Draw a large rectangle around all of the figures. Ask students to identify a symmetry line inside the rectangle. The reflection line is a line of symmetry between the two triangles. They might point out that there are other hidden symmetry lines in the sketch, such as those which cut perpendicularly through the sides of the framing rectangle, or within your triangles if they are not scalene.

Creating a Unit Circle Model

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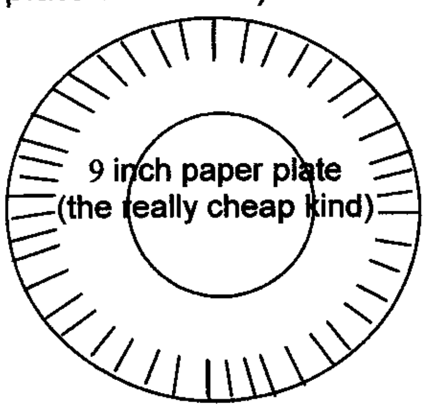
Paper Plate Unit Circle

The Unit Circle is a tool used in understanding sines and cosines of angles found in right triangles. It is so named because its radius is exactly one unit in length, usually just called "one". The circle's center is at the origin, and its circumference comprises the set of all points that are exactly one unit from the origin while lying in the plane.

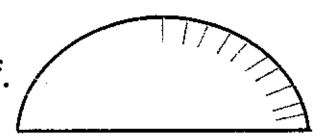
Materials: paper plates, wax sticks, card stock copies of special right triangles, markers, straight edges.

Directions

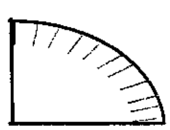
Step 1: Flatten the ridges around the outside of the plate (making the plate a flat disk).



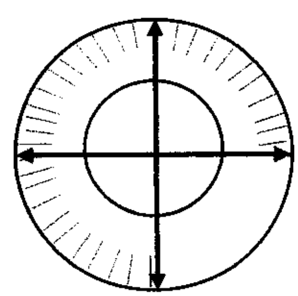
Step 2: Fold the plate in half.



Fold in half again. (Fourths)

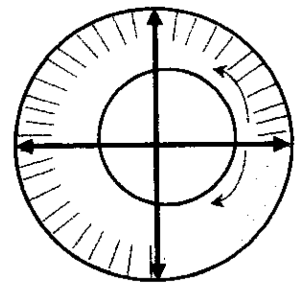


Step 3: Unfold and mark the x and y axes. Mark the right side of the x-axis darker. (Introduce the terms standard position, central angle, terminal side, initial side.



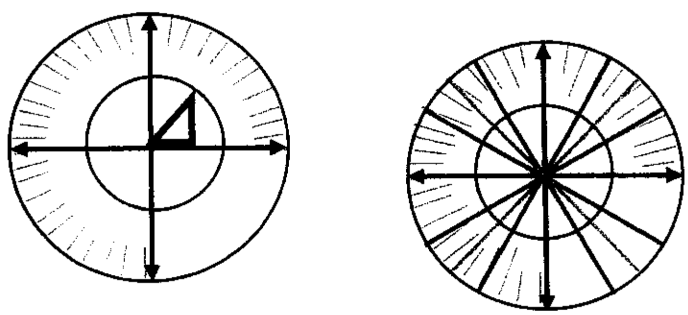
"Fill Up Your Plate" by V Meeks
Adapted from presentations by Sue Smith and Betty Mayberry

Step 4: Mark the direction and sign of angles. (Introduce terms clockwise and counterclockwise.



Step 5: Mark all quadrantal angles using one color. (Throughout the whole activity mark all families of angles with the same color)

Step 6: Mark the other special angles using the special right triangle cutouts. Measure the angles from the x-axis to set up later the definition of reference angles or triangles.



Step 7: Using a wax stick measure the radius of the smaller concentric circle. This length we call 1 radius length. Place the wax stick around the circumference of the circle starting on the initial side counterclockwise; mark the length of a radius all the way around as far as possible. Introduce the definition of radian measure.

Step 8: Discuss and develop with class the relation between radian and degree measure.

Step 9: Use the special right triangles, to mark the coordinates around the unit circle.

Graphing Sine and Cosine Parent Graphs

Materials:
10 - 15 pieces of spaghetti for each group
3 unit circles with thirds fourths and sixths marked off
Paper
Ruler
Glue

Directions:

1. Draw the y-axis on the left side of the short side of the paper.

   Mark off 16 "1" inch units along the x-axis and label them
   $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi$, *etc to* 2π.

2. Take a piece of spaghetti for each of the following and measure the distance of $\sin 0$, $\sin\frac{\pi}{6}$, $\sin\frac{\pi}{4}$, $\sin\frac{\pi}{3}$, $\sin\frac{\pi}{2}$, *etc* $\sin 2\pi$.

   You will break your spaghetti at the same length as the opposite leg of your triangle. (i.e. break your spaghetti at the distance from your angle value to the x-axis.)

3. Now take your spaghetti sticks and place them with the bottom on the appropriate x coordinate spots on the x-axis. Glue the spaghetti stick in place.

4. Continue your measurement and glueing the spaghetti sticks for each unit until you reach 2π

5. When you have completed gluing, use a marker and draw the curve that connects the tops of the spaghetti sticks. Congratulations, you have discovered the parent graph of the Sine function.

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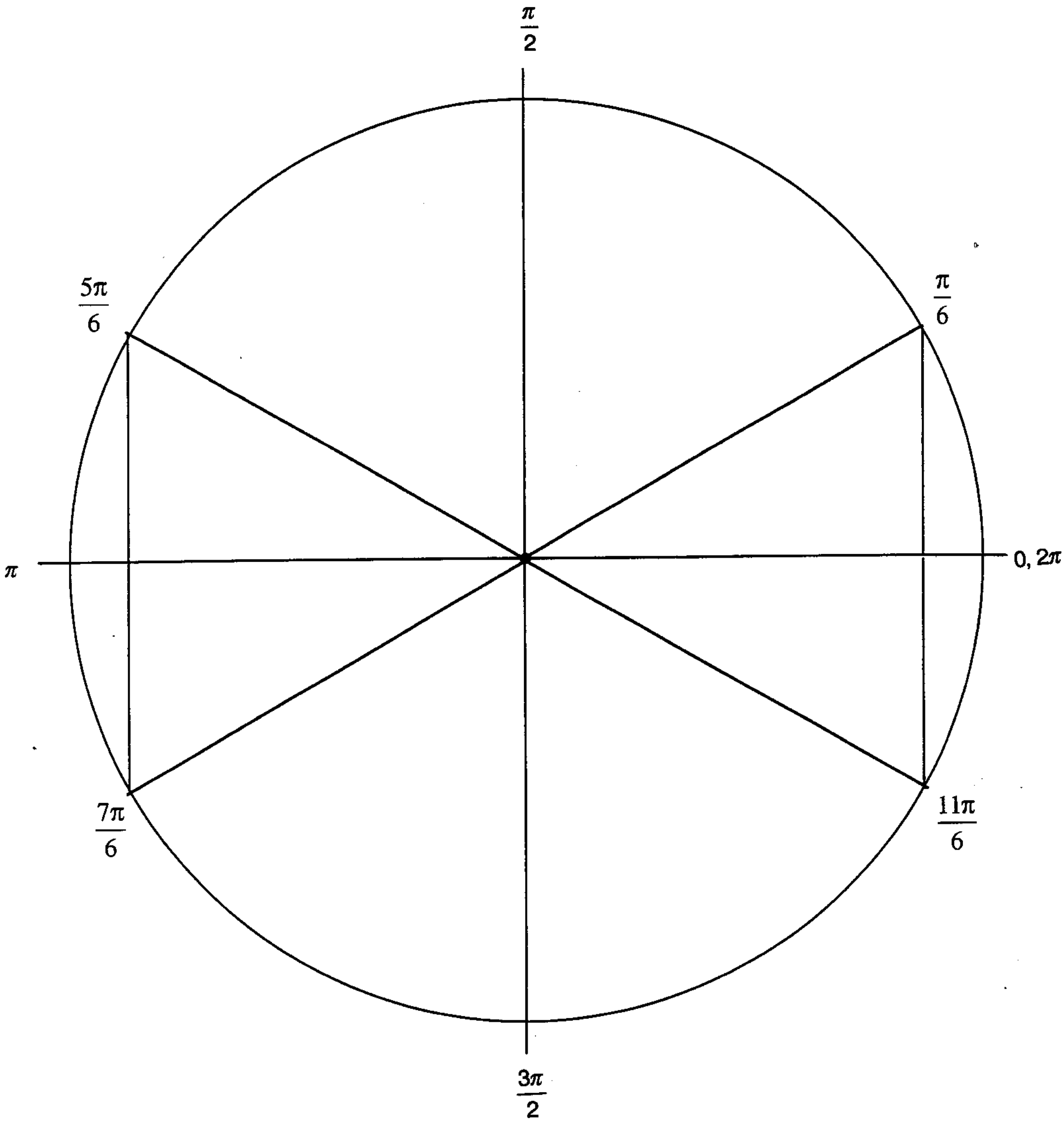
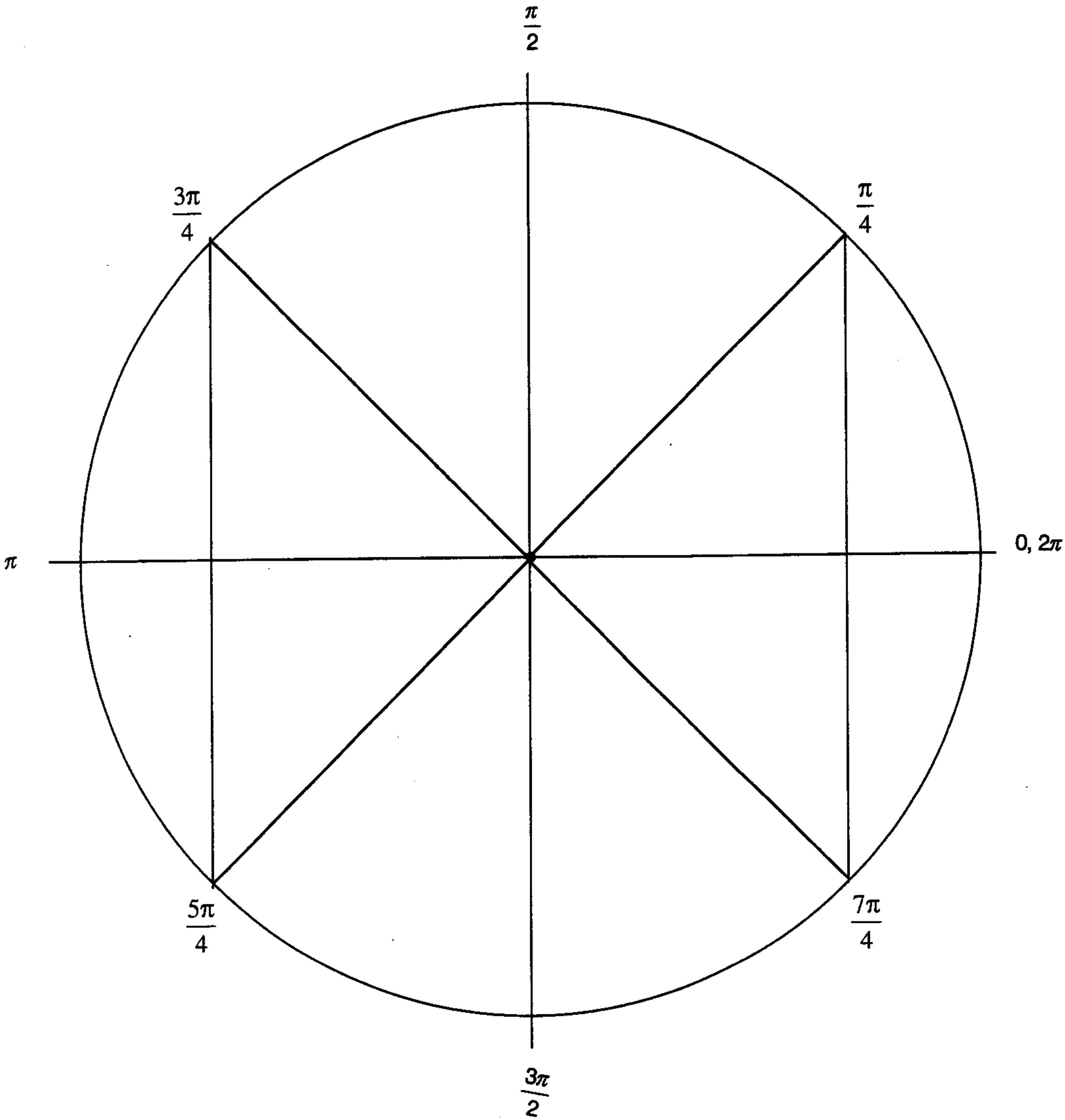


$\frac{\pi}{2}$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
π
$0, 2\pi$
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
$\frac{3\pi}{2}$

256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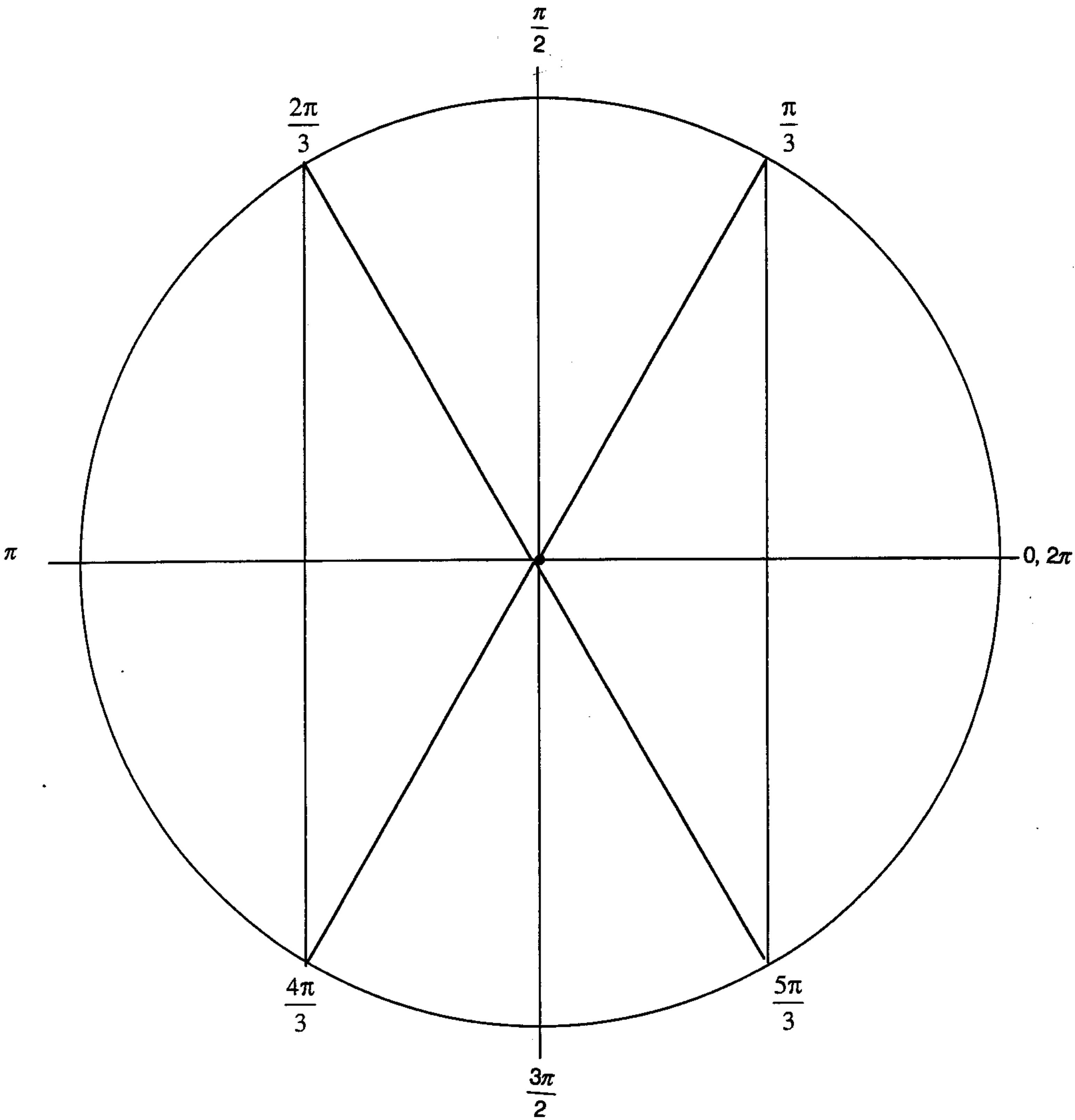
$\frac{3\pi}{2}$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$0, 2\pi$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
$\frac{3\pi}{2}$

Making the Unit Circle Game

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Materials needed

1. Garland

   A. 2 Strands of Color 1 for the $\frac{\pi}{3}$ B. 2 Strands of Color 2 for the $\frac{\pi}{4}$ C. 2 Strands of Color 3 for the $\frac{\pi}{6}$

   D. 2 Strands of Color 4 for the quadrantals

2. Construction paper with many colors

Preparation before playing game (All of these are suggestions for how I made my game. Any color scheme you choose or shapes will be fine)

To make the unit circle itself:

1. Make large yellow rectangles (1/2 sheets) with all of the radian measures (17 including 0 & 2π).
2. Make large blue triangles (1/2 sheet) with all of the degree measures. (17 including 0 & 360)
3. Make orange infinity shaped figures to put the coordinating points on (16)
4. Make a brightly colored star to be the origin.
5. Make large green triangles with roman numerals I-IV for the quadrants.

To make the pieces that allow the kids to understand and reinforce the circle:

6. Make 18 (9 per page) bright green rectangles with positive and negative π values. These will make the "Move it" part of the game. Make sure you include values that will go around the circle more than once and odd/even π.
7. Make 18 (9per page) purple rectangles with sin/cos/tan (radian measure found on unit circle) (e.g. $\cos\frac{5\pi}{6}$, $\tan-\frac{3\pi}{4}$). This will make the "Basic" cards for the game.
8. Make 18 (9 per page) pink rectangles with sec/csc/cot (radian measure found on unit circle) (e.g. $\csc\frac{\pi}{2}$, $\cot\frac{19\pi}{3}$). This will make the "Flip It!" cards for the game.
9. Make 18 (9 per page) red rectangles with trig equations whose solutions are basic unit circle radian measures. (e.g. $\sin x = \frac{\sqrt{3}}{2}, \tan x = -1$). These will make the "Backwards" cards.

Playing the game

1. Hand out all of the radian measures, the origin and the quadrants.
2. Ask the origin to make the middle of the circle and lay down her origin.
3. Ask the radian quadrantals to find the person who is directly across from him and take each end of the Color garland and form the quadrants with their strands of garland and to lay down their construction paper to indicate the radian measure.

4. Ask the $\frac{\pi}{4}$'s to find the person who is directly across from him and take each end of the Color 3 garland and to lay down their construction paper at the end of each strand to indicate the radian measure.

5. Ask the $\frac{\pi}{6}$'s to find the person who is directly across from him and take each end of the Color 2 garland and to lay down their construction paper at the end of each strand to indicate the radian measure.

6. Ask the $\frac{\pi}{3}$'s to find the person who is directly across from him and take each end of the Color 1 garland and to lay down their construction paper at the end of each strand to indicate the radian measure.

7. Ask the quadrants to label their respective quadrants.

You have now made your unit circle!!

8. Ask all of the degrees to now place their degree beside each equivalent radian.

10. Ask each point to place their point above the part of the circle that it corresponds with.

The rest of what you do with this is strictly up to the level of your students and how much time that you have left! I will explain each of the different cards.

Move It! The instructions for these cards are to have the student move counterclockwise, starting at 0 until they land on the given measure. This is a chance for the student to move around the big circle that you have just made. It is great if your cards include negative movement and "over rotation". It allows the kid to visualize where they are "landing" on the unit circle. If the movement would take too long (100π) that is a good time to discuss shortcuts into finding out where one would land.

Basics The instructions for these cards are to have students move to the radian measure on the circle and then give the indicated trig value. The points that have been placed at that measure should be helpful. If you do not want that help, you might want to remove the points before this part of the game.

Flip it! The instructions for these cards are to have students move to the radian measure on the circle and then give the indicated trig value. These are the same instructions for the Basics cards but are for the reciprocal functions. I separated them so that if you had not discussed reciprocal functions yet, you could do that part later.

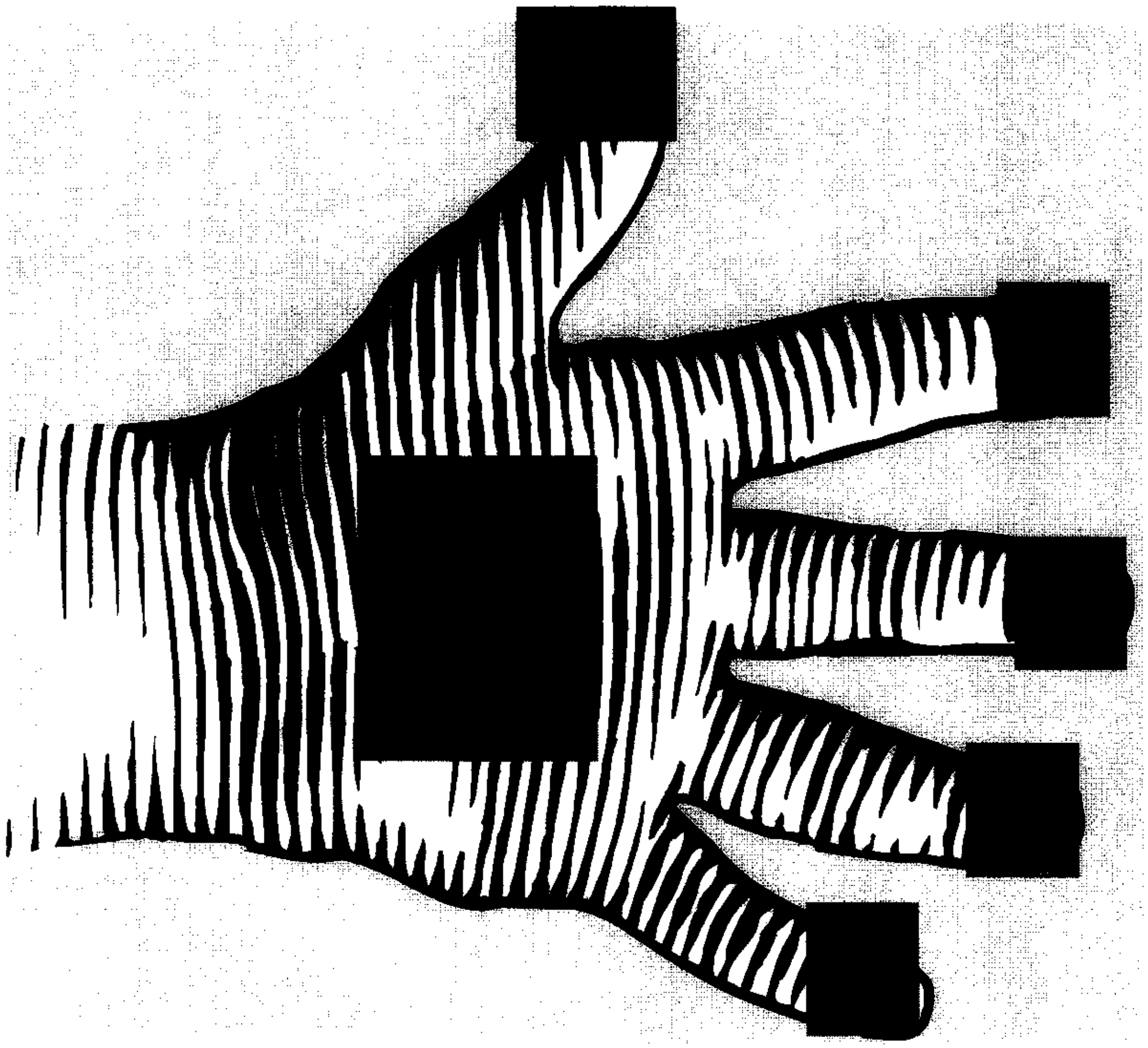
Backwards The instructions for these cards are to have students move to the radian measure on the circle where the answer to the equation is located. If two answers are possible, it is easy to see that on the big circle.

I hope that you have fun playing the game! Let me know if you have any questions. My e-mail address is above!

Mathematically yours,

Kitty Morgan

Your hand should look like this:



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ORGANIZER BY NUMBER

Unicycle	Tripod	Square	Cockroach	Bicycle	Spider
Partridge in a Pear Tree	Coins in a Fountain	Horsemen of the Apocalypse	Volleyball Team	Tango	Octave
Pogo Stick	Musketeers	Lucky Clover	Dice	Gemini	Stop Sign
Solo	Blind Mice	Tetrahedron	Hexagon	Duet	Octopus
Loneliest Number	Stooges	He's a Jolly Good Fellow	Geese a-Laying	Maids a-Milking	Turtle Doves

ORGANIZER BY COLOR

Mars	Lagoon	Lie	Medal	Grass	Coward	Tickled
Carpet	Bayou	Knight	Midas	Envy	Fever	Panther
Communist	Sad	Bleach	Rush	Thumb	Mellow	Shears
Cardinal	Bonnet	Snow	Nugget	Acres	Brick Road	Baby Girl
Slippers	Jay	Elephant	Bricker	Low on Experience	Jaundice	Flamingo